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Non-Isotropic Jacobi Spectral and Pseudospectral Methods in Three Dimensions

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Abstract. Non-isotropic Jacobi orthogonal approximation and Jacobi-Gauss type interpolation in three dimensions are investigated. The basic approximation results are established, which serve as the mathematical foundation of spectral and pseudospectral methods for singular problems, as well as problems defined on axisymmetric domains and some unbounded domains. The spectral and pseudospectral schemes are provided for two model problems. Their spectral accuracy is proved. Numerical results demonstrate the high efficiency of suggested algorithms.

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Key words: Jacobi spectral and pseudospectral methods in three dimensions, singular problems.

1 Introduction

The main advantage of spectral and pseudospectral methods is their high accuracy, see [3, 6–13]. In actual computations, the pseudospectral method is more preferable, since it only needs to evaluate unknown functions at interpolation nodes and so simplifies calculations. Moreover, it is much easier to deal with nonlinear terms. However, these merits may be destroyed by singularities of genuine solutions of considered problems, which could be caused by several factors, such as degenerating coefficients of derivatives of different orders involved in underlying problems. For solving such problems, Guo [16, 17], and Guo and Wang [24] developed the Jacobi orthogonal approximation and the Jacobi-Gauss type interpolation in non-uniformly weighted Sobolev space, and

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proposed the corresponding Jacobi spectral and pseudospectral methods with their applications to one-dimensional singular differential equations. We also refer to the work of [1, 11, 26]. The Jacobi spectral method is also useful for numerical solutions of differential equations defined on axisymmetric domains and some unbounded domains, see, e.g., [2, 14, 15, 18, 19, 31]. Besides, the Jacobi orthogonal approximation and Jacobi-Gauss type interpolation are related to various rational spectral and pseudospectral methods, see [4, 5, 10, 21, 22, 27, 28, 32]. Recently, Guo *et al.* [20], and Guo, *et al.* [23] studied the generalized Jacobi orthogonal and quasi-orthogonal approximations respectively, and so enlarge their applications. In practice, it is more important and interesting to solve the multi-dimensional singular problems and the related problems numerically. Guo and Wang [25] provided the Jacobi spectral method in two-dimensions, while Guo and Zhang [29] investigated the Jacobi pseudospectral method for two-dimensional problems. But there is few existing work dealing with three-dimensional problems by using the Jacobi spectral and pseudospectral methods.

This work is devoted to the Jacobi spectral and pseudospectral methods in three-dimensions. We first establish some results on the Jacobi orthogonal approximation, which play important role in designing and analyzing the Jacobi spectral method. Then, we consider the Jacobi-Gauss type interpolation, serving as the basic tool of the Jacobi pseudospectral method. We also derive a series of sharp results on the Legendre-Gauss type interpolation and the related Bernstein-Jackson type inequalities in three-dimensional space, which are very useful for pseudospectral method of partial differential equations with non-constant coefficients. As some applications of the above results, we provide the spectral and pseudospectral methods for a model problem. The numerical results demonstrate the high efficiency of the suggested algorithms and confirm the analysis well.

The paper is organized as follows. In the next section, we recall the basic results on the one-dimensional Jacobi orthogonal approximation and Jacobi-Gauss type interpolation. In Sections 3 and 4, we study the three-dimensional Jacobi orthogonal approximation and Jacobi-Gauss type interpolation, respectively. In Section 5, we propose the Jacobi spectral and pseudospectral methods for three-dimensional problems. In Section 6, we present some numerical results to demonstrate the efficiency of the proposed methods. The final Section is for some concluding remarks.

2 Preliminaries

2.1 Jacobi orthogonal approximation in one dimension

We now recall some results on the Jacobi orthogonal approximation in one dimension. Let $\Lambda=(-1,1)$ and $\alpha,\beta>-1$. The Jacobi polynomials of degree l are given by

$$(1-x)^{\alpha}(1+x)^{\beta}J_l^{(\alpha,\beta)}(x)=\frac{(-1)^l}{2^l l!}\partial_x^l\Big((1-x)^{l+\alpha}(1+x)^{l+\beta}\Big),\quad l=0,1,2\cdots.$$

Let $\chi^{(\alpha,\beta)}(x)=(1-x)^\alpha(1+x)^\beta$. The inner product and the norm of the weighted space $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ are denoted by $(u,v)_{\chi^{(\alpha,\beta)},\Lambda}$ and $\|v\|_{\chi^{(\alpha,\beta)},\Lambda}$, respectively. In particular, $\|v\|_\Lambda=\|v\|_{\chi^{(0,0)},\Lambda}$. The set of all Jacobi polynomials is a $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)-$orthogonal system, and

$$\|J_l^{(\alpha,\beta)}\|_{\chi^{(\alpha,\beta)},\Lambda}=\gamma_l^{(\alpha,\beta)}.$$

Let $\mathbb{N}$ be the set of all nonnegative integers. For any $N\in\mathbb{N}$. $\mathcal{P}_N(\Lambda)$ stands for the set of all algebraic polynomials of degree at most N. Further,

$${}_0\mathcal{P}_N(\Lambda)=\Big\{v\mid v\in\mathcal{P}_N(\Lambda),\,v(-1)=0\Big\},\quad \mathcal{P}_N^0(\Lambda)=\Big\{v\mid v\in{}_0\mathcal{P}_N(\Lambda),\,v(1)=0\Big\}.$$

The orthogonal projection $P_{N,\alpha,\beta,\Lambda}:L^2_{\chi^{(\alpha,\beta)}}(\Lambda)\to\mathcal{P}_N(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta,\Lambda}v-v,\phi)_{\chi^{(\alpha,\beta)},\Lambda}=0,\quad\forall\phi\in\mathcal{P}_N(\Lambda).$$

We denote by c a generic positive constant independent of any function and N. As a special case of Theorem 2.1 of [26] (also see [17]), we have that if $\partial_x^k v\in L^2_{\chi^{(\alpha+k,\beta+k)}}(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$, integers $0\le k\le r$ and $r\le N+1$, then

$$\|\partial_x^k(P_{N,\alpha,\beta,\Lambda}v-v)\|_{\chi^{(\alpha+k,\beta+k)},\Lambda}\le cN^{k-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}. \tag{2.1}$$

Next, we let $\alpha,\beta,\gamma,\delta>-1$, and define the space $H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$ with the norm

$$\|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}=\Big(\|\partial_x v\|^2_{\chi^{(\alpha,\beta)},\Lambda}+\|v\|^2_{\chi^{(\gamma,\delta)},\Lambda}\Big)^{\frac12}.$$

Moreover,

$$\begin{aligned}{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)&=\Big\{v\mid v\in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda),v(-1)=0\Big\},\\ H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)&=\Big\{v\mid v\in{}_0H^1_{\chi^{(\alpha,\beta)}}(\Lambda),\,v(1)=0\Big\}.\end{aligned}$$

We introduce the bilinear form

$$a_{\alpha,\beta,\gamma,\delta,\Lambda}(u,v)=(\partial_x u,\partial_x v)_{\chi^{(\alpha,\beta)},\Lambda}+(u,v)_{\chi^{(\gamma,\delta)},\Lambda},\quad\forall u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}:H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\to\mathcal{P}_N(\Lambda)$ is defined by

$$a_{\alpha,\beta,\gamma,\delta,\Lambda}(P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v,\phi)=0,\quad\forall\phi\in\mathcal{P}_N(\Lambda).$$

According to Theorem 3.1 of [26], we know that if $v\in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$, $\alpha\le\gamma+2,\beta\le\delta+2$ and integers $1\le r\le N+1$, then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}\le cN^{1-r}\|\partial_x^r v\|_{r,\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.2}$$

If, in addition, $\alpha\le\gamma+1$, $\beta\le\delta+1$, then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v\|_{\chi^{(\gamma,\delta)},\Lambda}\le cN^{-r}\|\partial_x^r v\|_{r,\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.3}$$

The orthogonal projection ${}_0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}:{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\to{}_0\mathcal{P}_N(\Lambda)$ is defined by

$$a_{\alpha,\beta,\gamma,\delta,\Lambda}({}_0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v,\phi)=0,\quad \forall\phi\in{}_0\mathcal{P}_N(\Lambda).$$

Thanks to Theorem 3.2 of [26], we assert that if $v\in{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$, $\alpha\le\gamma+2,\beta\le 0,\delta\ge 0$ (or $\alpha\le\gamma+1,\beta\le\delta+2,0<\alpha<1,\beta<1$), and integers $1\le r\le N+1$, then

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}\le cN^{1-r}\|\partial_x^r v\|_{r,\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.4}$$

If, in addition, $\alpha\le\gamma+1$, $\beta\le\delta+1$, then

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v\|_{\chi^{(\gamma,\delta)},\Lambda}\le cN^{-r}\|\partial_x^r v\|_{r,\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.5}$$

The orthogonal projection $\tilde{P}^{1,0}_{N,\alpha,\beta,\Lambda}:H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)\to\mathcal{P}^0_N(\Lambda)$ is defined by

$$(\partial_x(\tilde{P}^{1,0}_{N,\alpha,\beta,\Lambda}v-v),\partial_x\phi)_{\chi^{(\alpha,\beta)}}=0,\quad \forall\phi\in\mathcal{P}^0_N(\Lambda).$$

By the virtue of Theorem 3.4 of [26], we have that if $v\in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$, $-1<\alpha,\beta<1$, and integers $1\le r\le N+1$, then

$$\|\tilde{P}^{1,0}_{N,\alpha,\beta,\Lambda}v-v\|_{1,\chi^{(\alpha,\beta)},\Lambda}\le cN^{1-r}\|\partial_x^r v\|_{r,\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.6}$$

If, in addition, $-1<\alpha,\beta<0$ (or $0<\alpha,\beta<1$), then

$$\|\tilde{P}^{1,0}_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v\|_{\chi^{(\gamma,\delta)},\Lambda}\le cN^{-r}\|\partial_x^r v\|_{r,\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.7}$$

2.2 Jacobi-Gauss type interpolation in one dimension

We now turn to the Jacobi-Gauss type interpolation in one-dimension. Let $\zeta^{(\alpha,\beta)}_{G,N,j}$, $\zeta^{(\alpha,\beta)}_{R,N,j}$ and $\zeta^{(\alpha,\beta)}_{L,N,j}$, $0\le j\le N$, be the zeros of polynomials

$$J^{(\alpha,\beta)}_{N+1}(x),\ (1+x)J^{(\alpha,\beta+1)}_N(x),\ (1-x^2)\partial_x J^{(\alpha,\beta)}_N(x),$$

respectively. We denote by $\omega^{(\alpha,\beta)}_{Z,N,j}$, $0\le j\le N$, $Z=G,R,L$, the corresponding Christoffel numbers such that

$$\int_\Lambda \phi(x)\chi^{(\alpha,\beta)}(x)dx=\sum_{j=0}^N\phi(\zeta^{(\alpha,\beta)}_{Z,N,j})\omega^{(\alpha,\beta)}_{Z,N,j},\quad \forall\phi\in\mathcal{P}_{2N+\lambda_Z}(\Lambda), \tag{2.8}$$

where $\lambda_Z=1$ for $Z=G$, $\lambda_Z=0$ for $Z=R$, and $\lambda_Z=-1$ for $Z=L$, respectively. The corresponding discrete inner product and norm are as follows,

$$(u,v)_{\chi^{(\alpha,\beta)},Z,N,\Lambda}=\sum_{j=0}^{N}u(\zeta_{Z,N,j}^{(\alpha,\beta)})v(\zeta_{Z,N,j}^{(\alpha,\beta)})\omega_{Z,N,j}^{(\alpha,\beta)},$$
$$\|v\|_{\chi^{(\alpha,\beta)},Z,N,\Lambda}=(v,v)^{\frac{1}{2}}_{\chi^{(\alpha,\beta)},Z,N,\Lambda}.$$

By the exactness of (2.8), we have

$$(\phi,\psi)_{\chi^{(\alpha,\beta)},Z,N,\Lambda}=(\phi,\psi)_{\chi^{(\alpha,\beta)},\Lambda},\quad \forall\,\phi\psi\in P_{2N+\lambda_Z}(\Lambda). \tag{2.9}$$

Let $\Lambda_{Z,N}^{(\alpha,\beta)}=\{x\mid x=\zeta_{Z,N,j}^{(\alpha,\beta)},\,0\le j\le N\}$. We assume that $v\in C(\Lambda)$ for $Z=G$, $v\in C(\Lambda\cup\{x=-1\})$ for $Z=R$, and $v\in C(\bar{\Lambda})$ for $Z=L$. The Jacobi-Gauss type interpolation $I_{Z,N,\alpha,\beta,\Lambda}v\in\mathcal{P}_N(\Lambda)$ is determined uniquely by

$$I_{Z,N,\alpha,\beta,\Lambda}v(x)=v(x),\quad x\in\Lambda_{Z,N}^{(\alpha,\beta)}.$$

They are named as the Jacobi-Gauss interpolation for Z=G, the Jacobi-Gauss-Radau interpolation for Z=R, and the Jacobi-Gauss-Lobatto interpolation for Z=L, respectively.

According to Theorems 4.1, 4.5 and 4.9 of [26], we have the following results.

- If $v\in C(\Lambda)\cap L^2_{\chi^{(\alpha+k,\beta+l)}}(\Lambda)$, $\partial_x v\in L^2_{\chi^{(\alpha+k+1,\beta+l+1)}}(\Lambda)$, $k,l\in\mathbb{N}$ and $0\le k+l\le 1$, then

$$\|I_{G,N,\alpha,\beta,\Lambda}v\|_{\chi^{(\alpha+k,\beta+l)},\Lambda}\le c\|v\|_{\chi^{(\alpha+k,\beta+l)},\Lambda}+cN^{-1}\|\partial_x v\|_{\chi^{(\alpha+k+1,\beta+l+1)},\Lambda}. \tag{2.10}$$

- If $v\in C(\Lambda\cup\{-1\})\cap L^2_{\chi^{(\alpha+k,\beta-l)}}(\Lambda)$, $\partial_x v\in L^2_{\chi^{(\alpha+k+1,\beta-l+1)}}(\Lambda)$ with $v(-1)=0$, $k,l\in\mathbb{N}$ and $0\le k\le l\le 1$ and $l<\beta+1$, then

$$\|I_{R,N,\alpha,\beta,\Lambda}v\|_{\chi^{(\alpha+k,\beta-l)},\Lambda}\le c\|v\|_{\chi^{(\alpha+k,\beta-l)},\Lambda}+cN^{-1}\|\partial_x v\|_{\chi^{(\alpha+k+1,\beta-l+1)},\Lambda}. \tag{2.11}$$

- If $v\in C(\bar{\Lambda})\cap L^2_{\chi^{(\alpha-k,\beta-l)}}(\Lambda)$ with $v(\pm 1)=0$, $k,l\in\mathbb{N}$, $0\le k,l\le 1, k<\alpha+1$ and $l<\beta+1$, then

$$\|I_{L,N,\alpha,\beta,\Lambda}v\|_{\chi^{(\alpha-k,\beta-l)},\Lambda}\le c\|v\|_{\chi^{(\alpha-k,\beta-l)},\Lambda}+cN^{-1}\|\partial_x v\|_{\chi^{(\alpha-k+1,\beta-l+1)},\Lambda}. \tag{2.12}$$

Furthermore, as the special cases of Theorems 4.2, 4.6 and 4.10 of [26], we have the following approximation results.

- If $v\in C(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$, and integers $1\le r\le N+1$, then

$$\|I_{G,N,\alpha,\beta,\Lambda}v-v\|_{\chi^{(\alpha,\beta)},\Lambda}\le cN^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}. \tag{2.13}$$

- If $v\in C(\Lambda\cup\{-1\})$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$, and integers $1\le r\le N+1$, then

$$\|I_{R,N,\alpha,\beta,\Lambda}v-v\|_{\chi^{(\alpha,\beta)},\Lambda}\le cN^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)},\Lambda}. \tag{2.14}$$

- If $v\in C(\bar{\Lambda})$, $\partial_x^r v\in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}(\Lambda)}$, $-1<\alpha,\beta\le 0$ (or $0<\alpha,\beta\le 1$), and integers $1\le r\le N+1$, then

$$\|I_{L,N,\alpha,\beta,\Lambda}v-v\|_{\chi^{(\alpha,\beta)},\Lambda}\le cN^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)},\Lambda}. \tag{2.15}$$

We now focus on the Legendre-Gauss interpolation. The result (2.15) with $\alpha=\beta=0$ and $r=1$ implies

$$\|I_{L,N,0,0,\Lambda}v\|_\Lambda\le\|v\|_\Lambda+cN^{-1}\|\partial_x v\|_\Lambda. \tag{2.16}$$

Thanks to (2.16) of [29], we have that if $v\in H^1(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(r-1,r-1)}}(\Lambda)$ and integers $1\le r\le N+1$, then

$$\|\partial_x(I_{L,N,0,0,\Lambda}v-v)\|_\Lambda\le cN^{1-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}. \tag{2.17}$$

The above inequality with $r=1$ implies

$$\|\partial_x I_{L,N,0,0,\Lambda}v\|_\Lambda\le c\|\partial_x v\|_\Lambda. \tag{2.18}$$

According to (2.17) of [29], we have that if $v\in C(\bar{\Lambda})$, $\partial_x^r v\in L^2_{\chi^{(r-1,r-1)}}(\Lambda)$ and integers $1\le r\le N+1$, then

$$\|I_{L,N,0,0,\Lambda}v-v\|_{C(\Lambda)}\le cN^{\frac{1}{2}-r}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},\Lambda}. \tag{2.19}$$

This is a Bernstein-Jackson type inequality.

3 Jacobi orthogonal approximation in three dimensions

In this section, we study the non-isotropic Jacobi orthogonal approximation in three dimensions. We set $\Lambda_i=\{\,x_i\mid -1<x_i<1\}$, $i=1,2,3$, $\Omega=\Lambda_1\times\Lambda_2\times\Lambda_3$ and $x=(x_1,x_2,x_3)$. The inner product and norm of the weighted space $L^2_\chi(\Omega)$ are denoted by $(u,v)_{\chi,\Omega}$ and $\|u\|_{\chi,\Omega}$, respectively.

Let $\alpha_i,\bar{\alpha}_i,\tilde{\alpha}_i,\beta_i,\bar{\beta}_i,\tilde{\beta}_i,\gamma_i,\delta_i>-1$, for $i=1,2,3$, and

$$\alpha=(\alpha_1,\bar{\alpha}_1,\tilde{\alpha}_1,\alpha_2,\bar{\alpha}_2,\tilde{\alpha}_2,\alpha_3,\bar{\alpha}_3,\tilde{\alpha}_3),\qquad \beta=(\beta_1,\bar{\beta}_1,\tilde{\beta}_1,\beta_2,\bar{\beta}_2,\tilde{\beta}_2,\beta_3,\bar{\beta}_3,\tilde{\beta}_3),$$
$$\gamma=(\gamma_1,\gamma_2,\gamma_3),\qquad \delta=(\delta_1,\delta_2,\delta_3).$$

The three-dimensional Jacobi weight functions are as follows,

$$\begin{aligned}
\chi_1^{(\alpha,\beta)}(x)&=\chi^{(\alpha_1,\beta_1)}(x_1)\chi^{(\tilde{\alpha}_2,\tilde{\beta}_2)}(x_2)\chi^{(\bar{\alpha}_3,\bar{\beta}_3)}(x_3),\\
\chi_2^{(\alpha,\beta)}(x)&=\chi^{(\tilde{\alpha}_1,\tilde{\beta}_1)}(x_1)\chi^{(\alpha_2,\beta_2)}(x_2)\chi^{(\bar{\alpha}_3,\bar{\beta}_3)}(x_3),\\
\chi_3^{(\alpha,\beta)}(x)&=\chi^{(\bar{\alpha}_1,\bar{\beta}_1)}(x_1)\chi^{(\tilde{\alpha}_2,\tilde{\beta}_2)}(x_2)\chi^{(\alpha_3,\beta_3)}(x_3),\\
\chi^{(\gamma,\delta)}(x)&=\chi^{(\gamma_1,\delta_1)}(x_1)\chi^{(\gamma_2,\delta_2)}(x_2)\chi^{(\gamma_3,\delta_3)}(x_3).
\end{aligned}$$

We introduce the non-isotropic space $H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ with the following semi-norm and norm,

$$|v|_{1,\alpha,\beta,\Omega}=\left(\sum_{i=1}^{3}\|\partial_{x_i}v\|^2_{\chi_i^{(\alpha,\beta)},\Omega}\right)^{\frac{1}{2}},\quad \|v\|_{1,\alpha,\beta,\gamma,\delta,\Omega}=\left(|v|^2_{1,\alpha,\beta,\Omega}+\|v\|^2_{\chi^{(\gamma,\delta)},\Omega}\right)^{\frac{1}{2}}.$$

For $N=(N_1,N_2,N_3)\in\mathbb{N}^3$, $\mathcal{P}_N(\Omega)$ stands for the set of all algebraic polynomials of degree at most N_i with respect to x_i, $i=1,2,3$.

The $L^2_{\chi^{(\gamma,\delta)}}(\Omega)-$orthogonal projection $P_{N,\gamma,\delta,\Omega}:L^2_{\chi^{(\gamma,\delta)}}(\Omega)\to\mathcal{P}_N(\Omega)$, is defined by

$$(P_{N,\gamma,\delta,\Omega}v-v,\phi)_{\chi^{(\gamma,\delta)},\Omega}=0,\quad \forall\phi\in\mathcal{P}_N(\Omega).$$

In order to describe the approximation error, we let $r=(r_1,r_2,r_3)$, $r_i\in\mathbb{N}$, and

$$\begin{aligned}A_{r,\gamma,\delta}(v)=&\|\partial^{r_1}_{x_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r_1,\delta_1+r_1)}}(\Lambda_1)))}\\&+\|\partial^{r_2}_{x_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+r_2,\delta_2+r_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\&+\|\partial^{r_3}_{x_3}v\|_{L^2_{\chi^{(\gamma_3+r_3,\delta_3+r_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}.\end{aligned}$$

Theorem 3.1. *If $v\in L^2_{\chi^{(\gamma,\delta)}}(\Omega)$, integers $r_i\geq 0, r_i\leq N_i+1$ for $i=1,2,3$, and $A_{r,\gamma,\delta}(v)$ is finite, then*

$$\|P_{N,\gamma,\delta,\Omega}v-v\|_{\chi^{(\gamma,\delta)},\Omega}\leq c(N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3})A_{r,\gamma,\delta}(v).$$

Proof. Let $P_{N_i,\gamma_i,\delta_i,\Lambda_i}$, $i=1,2,3$, be the one-dimensional projections as in the last section. We use projection theorem and (2.1) with $k=0$ and $r=r_i$ or 0 to derive that

$$\begin{aligned}\|P_{N,\gamma,\delta,\Omega}v-v\|_{\chi^{(\gamma,\delta)},\Omega}=&\inf_{\phi\in\mathcal{P}_N(\Omega)}\|\phi-v\|_{\chi^{(\gamma,\delta)},\Omega}\\ \leq&\|P_{N_1,\gamma_1,\delta_1,\Lambda_1}v-v\|_{\chi^{(\gamma,\delta)},\Omega}+\|P_{N_1,\gamma_1,\delta_1,\Lambda_1}(P_{N_2,\gamma_2,\delta_2,\Lambda_2}v-v)\|_{\chi^{(\gamma,\delta)},\Omega}\\ \leq& c(N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3})A_{r,\gamma,\delta}(v).\end{aligned}$$

This completes the proof of the theorem. ☐

Next, let

$$a_{\alpha,\beta,\gamma,\delta,\Omega}(u,v)=\sum_{i=1}^{3}(\partial_{x_i}u,\partial_{x_i}v)_{\chi_i^{(\alpha,\beta)},\Omega}+(u,v)_{\chi^{(\gamma,\delta)},\Omega},\quad \forall u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega).$$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}:H^1_{\alpha,\beta,\gamma,\delta}(\Omega)\to\mathcal{P}_N(\Omega)$ is defined by

$$a_{\alpha,\beta,\gamma,\delta,\Omega}(P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}v-v,\phi)=0,\quad \forall\phi\in\mathcal{P}_N(\Omega).$$

We introduce the following quantity

$$\begin{aligned}
B_{r,\alpha,\beta,\gamma,\delta}(v)=&\|\partial_{x_1}^{r_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1+r_1-1,\beta_1+r_1-1)}}(\Lambda_1)))}\\
&+\|\partial_{x_2}^{r_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_2}^{r_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_2}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}^{r_1}\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1+r_1-1,\beta_1+r_1-1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}^{r_1}\partial_{x_3}v\|_{L^2_{\chi^{(\alpha_3,\beta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1+r_1-1,\beta_1+r_1-1)}}(\Lambda_1)))}\\
&+\|\partial_{x_2}^{r_2}\partial_{x_3}v\|_{L^2_{\chi^{(\alpha_3,\beta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}^{r_1}\partial_{x_2}\partial_{x_3}v\|_{L^2_{\chi^{(\alpha_3,\beta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1+r_1-1,\beta_1+r_1-1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_2}^{r_2}\partial_{x_3}v\|_{L^2_{\chi^{(\alpha_3,\beta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.
\end{aligned}$$

Theorem 3.2. *Let*

$$\gamma_i\le\min(\bar{\alpha}_i,\tilde{\alpha}_i),\quad \delta_i\le\min(\bar{\beta}_i,\tilde{\beta}_i),\quad i=1,2,3, \tag{3.1}$$

$$\alpha_i\le\gamma_i+2,\ \beta_i\le\delta_i+2,\quad i=1,2,3. \tag{3.2}$$

If $v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$, integers $1\le r_i\le N_i+1$ for $i=1,2,3$, and $B_{r,\alpha,\beta,\gamma,\delta}(v)$ is finite, then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}v-v\|_{1,\alpha,\beta,\gamma,\delta}\le c\left(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\right)B_{r,\alpha,\beta,\gamma,\delta}(v). \tag{3.3}$$

If in addition, if

$$\alpha_i\le\gamma_i+1,\ \beta_i\le\delta_i+1,\quad i=1,2,3, \tag{3.4}$$

then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}v-v\|_{\chi^{(\gamma,\delta)}}\le c\left(N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3}\right)B_{r,\alpha,\beta,\gamma,\delta}(v). \tag{3.5}$$

Proof. Let $P^1_{N_i,\alpha_i,\beta_i,\gamma_i,\delta_i,\Lambda_i}$, $i=1,2,3$, be the one-dimensional orthogonal projections as in the last section. We first consider the case $\alpha_i\le\gamma_i+2$, $\beta_i\le\delta_i+2$, $i=1,2,3$. According to the definition of the projection $P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}$, we have that

$$\|P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}v-v\|_{1,\alpha,\beta,\gamma,\delta,\Omega}\le W_1+W_2+W_3+W_4,$$

where

$$\begin{aligned}W_i&=\|\partial_{x_i}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v)\|_{\chi_i^{(\alpha,\beta)},\Omega},\quad i=1,2,3,\\ W_4&=\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v\|_{\chi^{(\gamma,\delta)},\Omega}.\end{aligned}$$

We have (cf. [26])

$$\max_{|x|\le1}(1-x)^a(1+x)^b=\begin{cases}1, & a=0,\quad b=0,\\ 2^{a+b}(\frac{a}{a+b})^a(\frac{b}{a+b})^b, & a,b\ge0,\ a^2+b^2\neq0.\end{cases}\tag{3.6}$$

Thus, we use (3.1) to obtain

$$\|\partial_{x_1}v\|_{\chi_1^{(\alpha,\beta)}}\le c\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.\tag{3.7}$$

Using (3.7) and (2.2) with $r=1$, we have that

$$\|\partial_{x_1}P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}v\|_{\chi_1^{(\alpha,\beta)}}\le c\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.\tag{3.8}$$

We now estimate W_1.

$$\begin{aligned}W_1\le&\|\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}v-v)\|_{\chi_1^{(\alpha,\beta)}}\\ &+\|\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}v-v))\|_{\chi_1^{(\alpha,\beta)}}\\ &+\|\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}(P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v)))\|_{\chi_1^{(\alpha,\beta)}}.\end{aligned}\tag{3.9}$$

By using (3.7) and (2.2), we derive that

$$\begin{aligned}&\|\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}v-v)\|_{\chi_1^{(\alpha,\beta)}}\\ \le&cN_1^{1-r_1}\|\partial_{x_1}^{r_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1+r_1-1,\beta_1+r_1-1)}}(\Lambda_1)))}.\end{aligned}\tag{3.10}$$

According to (3.8) and (2.2), we deduce that

$$\begin{aligned}&\|\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}v-v))\|_{\chi_1^{(\alpha,\beta)}}\\ \le&cN_2^{1-r_2}\|\partial_{x_2}^{r_2}\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.\end{aligned}\tag{3.11}$$

Using (3.8), (2.2) with $r=1$ and (2.2) with $r=r_3$, we derive that

$$\begin{aligned}&\|\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}(P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v)))\|_{\chi_1^{(\alpha,\beta)}}\\ \le{}& cN_3^{1-r_3}\|\partial_{x_1}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}\\ &+cN_3^{1-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.\end{aligned}\tag{3.12}$$

Substituting (3.10)-(3.12) into (3.9), we get that

$$W_1\le c\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\Big)\|v\|_{B_{r,\alpha,\beta,\gamma,\delta}(v)}.\tag{3.13}$$

Similarly, we can check that

$$W_2\le c\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\Big)\|v\|_{B_{r,\alpha,\beta,\gamma,\delta}(v)}.\tag{3.14}$$

$$W_3\le c\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\Big)\|v\|_{B_{r,\alpha,\beta,\gamma,\delta}(v)}.\tag{3.15}$$

Obviously,

$$\begin{aligned}W_4\le{}&\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}v-v\|_{\chi^{(\gamma,\delta)}}+\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}v-v)\|_{\chi^{(\gamma,\delta)}}\\ &+\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}(P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v))\|_{\chi^{(\gamma,\delta)}}.\end{aligned}\tag{3.16}$$

Thanks to (2.2), we have that

$$\begin{aligned}&\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}v-v\|_{\chi^{(\gamma,\delta)}}\\ \le{}& cN_1^{1-r_1}\|\partial_{x_1}^{r_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1+r_1-1,\beta_1+r_1-1)}}(\Lambda_1)))}.\end{aligned}\tag{3.17}$$

Using (2.2) with $r=1$, $r=r_2$ and $r=r_2-1$ successively, we get that

$$\begin{aligned}&\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}v-v)\|_{\chi^{(\gamma,\delta)}}\\ \le{}& cN_2^{1-r_2}\|\partial_{x_2}^{r_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\ &+cN_2^{1-r_2}\|\partial_{x_1}\partial_{x_2}^{r_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\alpha_2+r_2-1,\beta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.\end{aligned}\tag{3.18}$$

By using the same manner, we can deduce that

$$\begin{aligned}&\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}(P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v))\|_{\chi^{(\gamma,\delta)}}\\ \le{}& cN_3^{1-r_3}\|\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\ &+cN_3^{1-r_3}\|\partial_{x_2}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\ &+cN_3^{1-r_3}\|\partial_{x_1}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}\\ &+cN_3^{1-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\alpha_3+r_3-1,\beta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)))}.\end{aligned}\tag{3.19}$$

By substituting (3.17)-(3.19) into (3.16), we have that

$$W_4 \le c\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\Big)B_{r,\alpha,\beta,\gamma,\delta}(v). \tag{3.20}$$

A combination of (3.13)-(3.15) and (3.20) leads to the desired the result (3.3). We next consider the case with (3.4). Clearly,

$$\begin{aligned}
&\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}(P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v))\|_{\chi^{(\gamma,\delta)}}\\
\le&\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}v-v\|_{\chi^{(\gamma,\delta)}}+\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}v-v)\|_{\chi^{(\gamma,\delta)}}\\
&+\|P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1,\Lambda_1}P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2,\Lambda_2}(P^1_{N_3,\alpha_3,\beta_3,\gamma_3,\delta_3,\Lambda_3}v-v)\|_{\chi^{(\gamma,\delta)}}.
\end{aligned}$$

By (2.4) and the similar manner used above we can get (3.5) easily. □

Obviously, for $r_i=1$, $i=1,2,3$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}v-v\|_{1,\alpha,\beta,\gamma,\delta}\le\|v\|_{1,\alpha,\beta,\gamma,\delta}=B_{r,\alpha,\beta,\gamma,\delta}(v). \tag{3.21}$$

4 Jacobi-Gauss type interpolation in three dimensions

In this section, we study the Jacobi-Gauss type interpolation in three dimensions. Let $\mathbb{Z}=(Z_1,Z_2,Z_3)$, $Z_q=G,R,L$, $q=1,2,3$. The discrete inner product and norm in three dimensions are given by

$$(u,v)_{\chi^{(\gamma,\delta)},\mathbb{Z},N}=\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}\sum_{j_3=0}^{N_3}u(\zeta^{(\gamma,\delta)}_{\mathbb{Z},N,j})v(\zeta^{(\gamma,\delta)}_{\mathbb{Z},N,j})\omega^{(\gamma,\delta)}_{\mathbb{Z},N,j}, \tag{4.1}$$

$$\|v\|_{\chi^{(\gamma,\delta)},\mathbb{Z},N}=(v,v)^{\frac12}_{\chi^{(\gamma,\delta)},\mathbb{Z},N}, \tag{4.2}$$

where

$$\zeta^{(\gamma,\delta)}_{\mathbb{Z},N,j}=(\zeta^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1},\zeta^{(\gamma_2,\delta_2)}_{Z_2,N_2,j_2},\zeta^{(\gamma_3,\delta_3)}_{Z_3,N_3,j_3}),\quad \omega^{(\gamma,\delta)}_{\mathbb{Z},N,j}=\omega^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1}\omega^{(\gamma_2,\delta_2)}_{Z_2,N_2,j_2}\omega^{(\gamma_3,\delta_3)}_{Z_3,N_3,j_3}.$$

By the virtues of one-dimensional Jacobi discrete inner product, we have

$$(\phi_1\phi_2\phi_3,\psi_1\psi_2\psi_3)_{\chi^{(\gamma,\delta)},\mathbb{Z},N}=(\phi_1\phi_2\phi_3,\psi_1\psi_2\psi_3)_{\chi^{(\gamma,\delta)}},\quad \forall\ \phi_q\psi_q\in P_{2N_q+\lambda_{Z_q}},\ q=1,2,3. \tag{4.3}$$

Next, let

$$\Omega^{(\gamma,\delta)}_{\mathbb{Z},N}=\Big\{(x_1,x_2,x_3)\mid x_1=\zeta^{(\gamma_1,\delta_1)}_{Z_1,N_1,j_1},\ x_2=\zeta^{(\gamma_2,\delta_2)}_{Z_2,N_2,j_2},\ x_3=\zeta^{(\gamma_3,\delta_3)}_{Z_3,N_3,j_3},\ 0\le j_q\le N_q,\ q=1,2,3\Big\}.$$

Let $\mathbb{G}=(G,G,G)$, the three dimensional Jacobi-Gauss-type interpolation $I_{\mathbb{G},N,\gamma,\delta,\Omega}v(x)\in\mathcal{P}_N(\Omega)$, is determined uniquely by

$$I_{\mathbb{G},N,\gamma,\delta,\Omega}v(x)=v(x),\quad x\in\Omega^{(\gamma,\delta)}_{\mathbb{G},N}.$$

Let $k_q,l_q\in\mathbb{N}$, $q=1,2,3$, $k=(k_1,k_2,k_3)$, $l=(l_1,l_2,l_3)$ and

$$\chi^{(\gamma+k,\delta+l)}(x)=\chi^{(\gamma_1+k_1,\delta_1+l_1)}(x_1)\chi^{(\gamma_2+k_2,\delta_2+l_2)}(x_2)\chi^{(\gamma_3+k_3,\delta_3+l_3)}(x_3). \tag{4.4}$$

Now we introduce the non-isotropic spaces

$$\begin{aligned}M_{\mathbb{G},\gamma,\delta,k,l}(\Omega)=\Big\{\ v\mid\ & v\in L^2_{\chi^{(\gamma+k,\delta+l)}}(\Omega),\\
&\partial_{x_1}v\in L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1))),\\
&\partial_{x_2}v\in L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))),\\
&\partial_{x_3}v\in L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))),\\
&\partial_{x_1}\partial_{x_2}v\in L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1))),\\
&\partial_{x_1}\partial_{x_3}v\in L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1))),\\
&\partial_{x_2}\partial_{x_3}v\in L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1))),\\
&\partial_{x_1}\partial_{x_2}\partial_{x_3}v\in L^2_{\chi^{(\gamma+k+1,\delta+l+1)}}(\Omega)\Big\}.\end{aligned}$$

Equipped with the norm

$$\begin{aligned}C_{\gamma,\delta,k,l}(v)=&\|v\|_{L^2_{\chi^{(\gamma+k,\delta+l)}}(\Omega)}+\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)))}\\
&+\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_3}v\|_{L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_3}v\|_{L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)))}\\
&+\|\partial_{x_2}\partial_{x_3}v\|_{L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_2}\partial_{x_3}v\|_{L^2_{\chi^{(\gamma+k+1,\delta+l+1)}}(\Omega)}.\end{aligned}$$

Theorem 4.1. *For any* $v\in C(\Omega)\cap M_{\mathbb{G},\gamma,\delta,k,l}(\Omega)$, $k_q,l_q\in\mathbb{N}$ *and* $0\le k_q+l_q\le 1$, $q=1,2,3$,

$$\begin{aligned}&\|I_{\mathbb{G},N,\gamma,\delta,\Omega}v\|_{\chi^{(\gamma+k,\delta+l)},\Omega}\\
&\le c\Big(1+N_1^{-1}+N_2^{-1}+N_3^{-1}+N_1^{-1}N_2^{-1}+N_1^{-1}N_3^{-1}+N_2^{-1}N_3^{-1}\\
&\qquad+N_1^{-1}N_2^{-1}N_3^{-1}\Big)C_{\gamma,\delta,k,l}(v).\end{aligned} \tag{4.5}$$

Proof. Let $I_{G,N_i,\gamma_i,\delta_i,\Lambda_i}$ be the one-dimensional Jacobi-Gauss-type interpolation. Clearly, $I_{\mathbb{G},N,\gamma,\delta,\Omega}v = I_{G,N_3,\gamma_3,\delta_3,\Lambda_3} I_{G,N_2,\gamma_2,\delta_2,\Lambda_2} I_{G,N_1,\gamma_1,\delta_1,\Lambda_1} v$. Due to (2.10), we can verify that

$$
\begin{aligned}
&\|I_{\mathbb{G},N,\gamma,\delta,\Omega}v\|_{\chi^{(\gamma+k,\delta+l)},\Omega} \le c\|v\|_{L^2_{\chi^{(\gamma+k,\delta+l)}}(\Omega)}\\
&\quad + cN_1^{-1}\|\partial_{x_1}v\|_{L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)))}\\
&\quad + cN_2^{-1}\|\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)))}\\
&\quad + cN_3^{-1}\|\partial_{x_3}v\|_{L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)))}\\
&\quad + cN_1^{-1}N_2^{-1}\|\partial_{x_1}\partial_{x_2}v\|_{L^2_{\chi^{(\gamma_3+k_3,\delta_3+l_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)))}\\
&\quad + cN_1^{-1}N_3^{-1}\|\partial_{x_1}\partial_{x_3}v\|_{L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2,\delta_2+l_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1+1,\delta_1+l_1+1)}}(\Lambda_1)))}\\
&\quad + cN_2^{-1}N_3^{-1}\|\partial_{x_2}\partial_{x_3}v\|_{L^2_{\chi^{(\gamma_3+k_3+1,\delta_3+l_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+k_2+1,\delta_2+l_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+k_1,\delta_1+l_1)}}(\Lambda_1)))}\\
&\quad + cN_1^{-1}N_2^{-1}N_3^{-1}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}v\|_{L^2_{\chi^{(\gamma+k+1,\delta+l+1)}}(\Omega)}.
\end{aligned}
$$

This leads to the desired result (4.5). □

Remark 4.1. As to the Jacobi-Gauss-Radau interpolation $I_{\mathbb{R},N,\gamma,\delta,\Omega}v$ and the Jacobi-Gauss-Lobatto interpolation $I_{\mathbb{L},N,\gamma,\delta,\Omega}v$, we can also define the spaces as $M_{\mathbb{G},\gamma,\delta,k,l}(\Omega)$ with the norms as $C_{\gamma,\delta,k,l}(v)$, and using the same manner to get similar results as (4.5).

Next, we present the some results on the Jacobi-Gauss type interpolation. To do this, we introduce the non-isotropic spaces

$$
\begin{aligned}
&N_{r,\gamma,\delta}(\Omega)\\
=&\Big\{v\in L^2_{\chi^{(\gamma,\delta)}}(\Omega) \mid \partial_{x_1}^{r_1}v\in L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r_1,\delta_1+r_1)}}(\Lambda_1)))\\
&\quad \partial_{x_2}^{r_2}v\in L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+r_2,\delta_2+r_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))\\
&\quad \partial_{x_3}^{r_3}v\in L^2_{\chi^{(\gamma_3+r_3,\delta_3+r_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))\\
&\quad \partial_{x_1}\partial_{x_2}^{r_2-1}v\in L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+r_2-1,\delta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)))\\
&\quad \partial_{x_1}\partial_{x_3}^{r_3-1}v\in L^2_{\chi^{(\gamma_3+r_3-1,\delta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)))\\
&\quad \partial_{x_2}\partial_{x_3}^{r_3-1}v\in L^2_{\chi^{(\gamma_3+r_3-1,\delta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+1,\delta_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))\\
&\quad \partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-2}v\in L^2_{\chi^{(\gamma_3+r_3-2,\delta_3+r_3-2)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+1,\delta_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)))\Big\},
\end{aligned}
$$

equipped with the semi-norm

$$\begin{aligned}D_{r,\gamma,\delta}(v)=&\|\partial_{x_1}^{r_1}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+r_1,\delta_1+r_1)}}(\Lambda_1)))}\\&+\|\partial_{x_2}^{r_2}v\|_{L^2_{\chi^{(\gamma_3,\delta_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+r_2,\delta_2+r_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\&+\|\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(\gamma_3+r_3,\delta_3+r_3)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\&+\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{L^2_{\chi^{(\gamma_3+1,\delta_3+1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+r_2-1,\delta_2+r_2-1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1}}(\Lambda_1)))}\\&+\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(\gamma_3+r_3-1,\delta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2,\delta_2)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)))}\\&+\|\partial_{x_2}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(\gamma_3+r_3-1,\delta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+1,\delta_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1,\delta_1)}}(\Lambda_1)))}\\&+\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(\gamma_3+r_3-1,\delta_3+r_3-1)}}(\Lambda_3;L^2_{\chi^{(\gamma_2+1,\delta_2+1)}}(\Lambda_2;L^2_{\chi^{(\gamma_1+1,\delta_1+1)}}(\Lambda_1)))},\end{aligned}$$

and the norm

$$\|v\|_{N_{r,\gamma,\delta}(\Omega)}=\|v\|_{\chi^{(\gamma,\delta)},\Omega}+D_{r,\gamma,\delta}(v).$$

Similarly, by using (2.10) and (2.13) we can get the following result.

Theorem 4.2. *For any $v\in C(\Omega)\cap N_{r,\gamma,\delta}(\Omega)$ and integer $2\le r_i\le N_i+1$, $i=1,2,3$, then,*

$$\begin{aligned}&\|I_{\mathbb{G},N,\gamma,\delta,\Omega}v-v\|_{\chi^{(\gamma,\delta)},\Omega}\\&\le c\Big(N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3}+N_1^{-1}N_2^{1-r_2}+N_1^{-1}N_3^{1-r_3}\\&\qquad+N_2^{-1}N_3^{1-r_3}+N_1^{-1}N_2^{-1}N_3^{2-r_3}\Big)D_{r,\gamma,\delta}(v).\end{aligned}\tag{4.6}$$

Remark 4.2. If in addition $N_1=\mathcal{O}(N_2)=\mathcal{O}(N_3)$ in Theorem 4.2, then result (4.6) can be written as

$$\|I_{\mathbb{G},N,\gamma,\delta,\Omega}v-v\|_{\chi^{(\gamma,\delta)},\Omega}\le c\Big(N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3}\Big)D_{r,\gamma,\delta}(v).\tag{4.7}$$

Moreover, as a special case we consider the three-dimensional Legendre-Gauss-Lobatto interpolation and related Bernstein-Jackson-type inequalities, which will be used in the sequel.

Let $\mathbb{L}=(L,L,L)$, for $v\in C(\bar{\Omega})$, we define the Legendre-Gauss-Lobatto interpolation as

$$I_{\mathbb{L},N,0,0,\Omega}v=I_{L,N_1,0,0,\Lambda_1}I_{L,N_2,0,0,\Lambda_2}I_{L,N_3,0,0,\Lambda_3}v.$$

In order to describe the error of Legendre-Gauss-Lobatto interpolation we define the following quantity with $r_i \geq 2$, $i=1,2,3$.

$$\begin{aligned}
E_r(v)=\Big(&\|\partial_{x_1}^{r_1}v\|^2_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))}+\|\partial_{x_2}^{r_2}v\|^2_{L^2(\Lambda_3;L^2_{\chi^{(r_2-1,r_2-1)}}(\Lambda_2;L^2(\Lambda_1)))}\\
&+\|\partial_{x_3}^{r_3}v\|^2_{L^2_{\chi^{(r_3-1,r_3-1)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}+\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|^2_{L^2(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|^2_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}+\|\partial_{x_2}\partial_{x_3}^{r_3-1}v\|^2_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}\\
&+\|\partial_{x_1}^{r_1-1}\partial_{x_2}v\|^2_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-2,r_1-2)}}(\Lambda_1)))}+\|\partial_{x_1}^{r_1-1}\partial_{x_3}v\|^2_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-2,r_1-2)}}(\Lambda_1)))}\\
&+\|\partial_{x_2}^{r_2-1}\partial_{x_3}v\|^2_{L^2(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))}+\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-2}v\|^2_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}\\
&+\|\partial_{x_1}\partial_{x_2}^{r_2-2}\partial_{x_3}v\|^2_{L^2(\Lambda_3;L^2_{\chi^{(r_2-3,r_3-3)}}(\Lambda_2;L^2(\Lambda_1)))}+\|\partial_{x_1}^{r_1-2}\partial_{x_2}\partial_{x_3}v\|^2_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-3,r_1-3)}}(\Lambda_1)))}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Theorem 4.3. If $E_r(v)$ is finite. Then for any $v\in C(\overline{\Omega})$ and $2\leq r_i\leq N_i+1$, $i=1,2,3$,

$$\begin{aligned}
\|I_{\mathbb{L},N,0,0,\Omega}v-v\|_\Omega\leq c\Big(&N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3}+N_1^{-1}N_2^{1-r_2}\\
&+N_1^{-1}N_3^{1-r_3}+N_2^{-1}N_3^{1-r_3}+N_1^{-1}N_2^{-1}N_3^{2-r_3}\Big)E_r(v). \quad (4.8)
\end{aligned}$$

If in addition $N_1=\mathcal{O}(N_2)=\mathcal{O}(N_3)$, then

$$\|I_{\mathbb{L},N,0,0,\Omega}v-v\|_\Omega\leq c\Big(N_1^{-r_1}+N_2^{-r_2}+N_3^{-r_3}\Big)E_r(v). \quad (4.9)$$

Proof. According to the definition of $I_{\mathbb{L},N,0,0,\Omega}$, we have that

$$\begin{aligned}
&\|I_{\mathbb{L},N,0,0,\Omega}v-v\|_\Omega\\
\leq&\|I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}(I_{L,N_3,0,0,\Lambda_3}v-v))\|_\Omega \quad (4.10)\\
&+\|I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}v-v)\|_\Omega+\|I_{L,N_1,0,0,\Lambda_1}v-v\|_\Omega.
\end{aligned}$$

Due to (2.15) with $\alpha=\beta=0$, we get that

$$\|I_{L,N_1,0,0,\Lambda_1}v-v\|_\Omega\leq cN_1^{-r_1}\|\partial_{x_1}^{r_1}v\|_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))}. \quad (4.11)$$

By using (2.16) and (2.15) with $\alpha=\beta=0$, we derive that

$$\begin{aligned}
&\|I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}v-v)\|_\Omega\\
\leq&cN_2^{-r_2}\|\partial_{x_2}^{r_2}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-1,r_2-1)}}(\Lambda_2;L^2(\Lambda_1)))}\\
&+cN_1^{-1}N_2^{1-r_2}\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))}. \quad (4.12)
\end{aligned}$$

According to (2.18), (2.16) and (2.15) with $\alpha=\beta=0$, we deduce that

$$\begin{aligned}&\|I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}(I_{L,N_3,0,0,\Lambda_3}v-v))\|_\Omega\\ \le& cN_3^{-r_3}\|\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(r_3-1,r_3-1)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}\\ &+cN_1^{-1}N_3^{1-r_3}\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}\\ &+cN_2^{-1}N_3^{1-r_3}\|\partial_{x_2}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}\\ &+cN_1^{-1}N_2^{-1}N_3^{2-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-2}v\|_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}.\end{aligned}\tag{4.13}$$

Substituting (4.11), (4.12) and (4.13) into (4.10) we get the desired result (4.8). If $N_1=\mathcal{O}(N_2)=\mathcal{O}(N_3)$, the result (4.9) is clear. □

Theorem 4.4. If $E_r(v)$ is finite. Then for any $v\in C(\overline{\Omega})$ and $2\le r_i\le N_i+1$, $i=1,2,3$,

$$\begin{aligned}|I_{\mathbb{L},N,0,0,\Omega}v-v|_{H^1(\Omega)}\le c\Big(&N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\\ &+N_1^{-1}N_2^{2-r_2}+N_2^{-1}N_3^{2-r_3}+N_3^{-1}N_1^{2-r_1}\Big)E_r(v).\end{aligned}\tag{4.14}$$

If in addition $N_1=\mathcal{O}(N_2)=\mathcal{O}(N_3)$, then

$$|I_{\mathbb{L},N,0,0,\Omega}v-v|_{H^1(\Omega)}\le c\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\Big)E_r(v).\tag{4.15}$$

Proof. Obviously,

$$\begin{aligned}&\|\partial_{x_1}(I_{\mathbb{L},N,0,0,\Omega}v-v)\|_\Omega\\ \le&\|\partial_{x_1}(I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}(I_{L,N_3,0,0,\Lambda_3}v-v)))\|_\Omega\\ &+\|\partial_{x_1}(I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}v-v))\|_\Omega+\|\partial_{x_1}(I_{L,N_1,0,0,\Lambda_1}v-v)\|_\Omega.\end{aligned}\tag{4.16}$$

By (2.17), we have that

$$\|\partial_{x_1}(I_{L,N_1,0,0,\Lambda_1}v-v)\|_\Omega\le cN_1^{1-r_1}\|\partial_{x_1}^{r_1}v\|_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))}.\tag{4.17}$$

Thanks to (2.18) and (2.15), we deduce that

$$\begin{aligned}&\|\partial_{x_1}(I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}v-v))\|_\Omega\\ \le& cN_2^{1-r_2}\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))}.\end{aligned}\tag{4.18}$$

Using (2.18), (2.16) and (2.15) successively, we derive that

$$\begin{aligned}
&\|\partial_{x_1}(I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}(I_{L,N_3,0,0,\Lambda_3}v-v)))\|_\Omega \\
\le& cN_3^{1-r_3}\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_2^{-1}N_3^{2-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-2}v\|_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}. \qquad (4.19)
\end{aligned}$$

Substituting (4.17)-(4.19) into (4.16) yields

$$\begin{aligned}
\|\partial_{x_1}(I_{\mathbb{L},N,0,0,\Omega}v-v)\|_\Omega \le& cN_1^{1-r_1}\|\partial_{x_1}^{r_1}v\|_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))} \\
&+cN_2^{1-r_2}\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_3^{1-r_3}\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_2^{-1}N_3^{2-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-2}v\|_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}. \qquad (4.20)
\end{aligned}$$

Similarly, we can get the following two estimates

$$\begin{aligned}
\|\partial_{x_2}(I_{\mathbb{L},N,0,0,\Omega}v-v)\|_\Omega \le& cN_2^{1-r_2}\|\partial_{x_2}^{r_2}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-1,r_2-1)}}(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_1^{1-r_2}\|\partial_{x_2}\partial_{x_1}^{r_1-1}v\|_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-2,r_1-2)}}(\Lambda_1)))} \\
&+cN_3^{1-r_3}\|\partial_{x_2}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_1^{-1}N_3^{2-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-2}v\|_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}. \qquad (4.21)
\end{aligned}$$

$$\begin{aligned}
\|\partial_{x_3}(I_{\mathbb{L},N,0,0,\Omega}v-v)\|_\Omega \le& cN_3^{1-r_3}\|\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(r_3-1,r_3-1)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_1^{1-r_1}\|\partial_{x_3}\partial_{x_1}^{r_1-1}v\|_{L^2(\Lambda_3;L^2(\Lambda_2;L^2_{\chi^{(r_1-2,r_1-2)}}(\Lambda_1)))} \\
&+cN_2^{1-r_3}\|\partial_{x_3}\partial_{x_2}^{r_2-1}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))} \\
&+cN_1^{-1}N_2^{2-r_2}\|\partial_{x_1}\partial_{x_3}\partial_{x_2}^{r_3-2}v\|_{L^2(\Lambda_3;L^2_{\chi^{(r_2-3,r_2-3)}}(\Lambda_2;L^2(\Lambda_1)))}. \qquad (4.22)
\end{aligned}$$

A combination of (4.20)-(4.22) leads to the desired result (4.14). This with $N_1=\mathcal{O}(N_2)=\mathcal{O}(N_3)$ implies (4.15). □

Finally, we consider the Berstein-Jackson type inequality. Let $\eta=(\eta_1,\eta_2,\eta_3)$, $0\le\eta_i\le\frac{1}{2}$, $i=1,2,3$, and $r_i>2$, $i=1,2,3$. We denote the quantity

$$\begin{aligned}&|v|_{F^{r,\eta}(\Omega)}\\ =&\Big(\|\partial_{x_1}^{r_1}v\|^2_{H^{\frac12+\eta_3}(\Lambda_3;H^{\frac12+\eta_2}(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))}+\|\partial_{x_2}^{r_2}v\|^2_{H^{\frac12+\eta_3}(\Lambda_3;L^2_{\chi^{(r_2-1,r_2-1)}}(\Lambda_2;H^{\frac12+\eta_1}(\Lambda_1)))}\\ &+\|\partial_{x_3}^{r_3}v\|^2_{L^2_{\chi^{(r_3-1,r_3-1)}}(\Lambda_3;H^{\frac12+\eta_2}(\Lambda_2;H^{\frac12+\eta_1}(\Lambda_1)))}+\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|^2_{H^{\frac12+\eta_3}(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))}\\ &+\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|^2_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;H^{\frac12+\eta_2}(\Lambda_2;L^2(\Lambda_1)))}+\|\partial_{x_2}\partial_{x_3}^{r_3-1}v\|^2_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;H^{\frac12+\eta_2}(\Lambda_1)))}\\ &+\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-1}v\|^2_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}\Big)^{\frac12},\end{aligned}$$

and

$$\|v\|_{F^{r,\eta}(\Omega)}=(\|v\|^2_{C(\Omega)}+|v|^2_{F^{r,\eta}(\Omega)})^{\frac12}.$$

Obviously,

$$\begin{aligned}&\|I_{\mathbb{L},N,0,0,\Omega}v-v\|_{C(\Omega)}\\ \le&\|I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}(I_{L,N_3,0,0,\Lambda_3}v-v))\|_{C(\Omega)}\\ &+\|(I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}v-v)\|_{C(\Omega)}+\|(I_{L,N_1,0,0,\Lambda_1}-I)v\|_{C(\Omega)}.\end{aligned}\tag{4.23}$$

Thanks to (2.19) and the embedding inequality, we have that

$$\begin{aligned}\|I_{L,N_1,0,0,\Lambda_1}v-v\|_{C(\Omega)}\le& cN_1^{\frac12-r_1}\|\partial_{x_1}^{r_1}v\|_{C(\Lambda_3;C(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))}\\ \le& cN_1^{\frac12-r_1}\|\partial_{x_1}^{r_1}v\|_{H^{\frac12+\eta_3}(\Lambda_3;H^{\frac12+\eta_2}(\Lambda_2;L^2_{\chi^{(r_1-1,r_1-1)}}(\Lambda_1)))}.\end{aligned}\tag{4.24}$$

According to (2.19) with $r=1$, $r=r_2$, $r=r_2-1$ successively, and the embedding inequality, we get that

$$\begin{aligned}&\|(I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}v-v)\|_{C(\Omega)}\\ \le& cN_2^{\frac12-r_2}\|\partial_{x_2}^{r_2}v\|_{H^{\frac12+\eta_3}(\Lambda_3;L^2_{\chi^{(r_2-1,r_2-1)}}(\Lambda_2;H^{\frac12+\eta_1}(\Lambda_1)))}\\ &+cN_1^{-\frac12}N_2^{\frac32-r_2}\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{H^{\frac12+\eta_3}(\Lambda_3;L^2_{\chi^{(r_2-2,r_2-2)}}(\Lambda_2;L^2(\Lambda_1)))}.\end{aligned}\tag{4.25}$$

Using (2.19) with $r=1$ repeatedly, (2.19) with $r=r_3$, $r=r_3-1$ and the embedding inequality, we derive that

$$\begin{aligned}
&\|I_{L,N_1,0,0,\Lambda_1}(I_{L,N_2,0,0,\Lambda_2}(I_{L,N_3,0,0,\Lambda_3}v-v))\|_{C(\Omega)}\\
\le& cN_3^{\frac{1}{2}-r_3}\|\partial_{x_3}^{r_3}v\|_{L^2_{\chi^{(r_3-1,r_3-1)}}(\Lambda_3;H^{\frac{1}{2}+\eta_2}(\Lambda_2;H^{\frac{1}{2}+\eta_1}(\Lambda_1;)))}\\
&+cN_2^{-\frac{1}{2}}N_3^{\frac{3}{2}-r_3}\|\partial_{x_2}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;L^2(\Lambda_2;H^{\frac{1}{2}+\eta_1}(\Lambda_1)))}\\
&+cN_1^{-\frac{1}{2}}N_3^{\frac{3}{2}-r_3}\|\partial_{x_1}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-2,r_3-2)}}(\Lambda_3;H^{\frac{1}{2}+\eta_2}(\Lambda_2;L^2(\Lambda_1)))}\\
&+cN_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{\frac{3}{2}-r_3}\|\partial_{x_1}\partial_{x_2}\partial_{x_3}^{r_3-1}v\|_{L^2_{\chi^{(r_3-3,r_3-3)}}(\Lambda_3;L^2(\Lambda_2;L^2(\Lambda_1)))}. \qquad (4.26)
\end{aligned}$$

By substituting the above three inequalities into (4.23), we reach that

$$\begin{aligned}
\|I_{\mathbb{L},N,0,0,\Omega}v-v\|_{C(\Omega)}\le\Big(&N_1^{\frac{1}{2}-r_1}+N_2^{\frac{1}{2}-r_2}+N_3^{\frac{1}{2}-r_3}+N_1^{-\frac{1}{2}}N_2^{\frac{3}{2}-r_2}+cN_1^{-\frac{1}{2}}N_3^{\frac{3}{2}-r_3}\\
&+N_2^{-\frac{1}{2}}N_3^{\frac{3}{2}-r_3}+N_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{\frac{3}{2}-r_3}\Big)|v|_{F^{r,\eta}(\Omega)}. \qquad (4.27)
\end{aligned}$$

If $N_1=\mathcal{O}(N_2)=\mathcal{O}(N_3)$, then the above estimation can be written as

$$\|I_{\mathbb{L},N,0,0,\Omega}v-v\|_{C(\Omega)}\le\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\Big)|v|_{F^{r,\eta}(\Omega)}. \qquad (4.28)$$

5 Jacobi method for three dimensional singular problem

This section is devoted to the Jacobi spectral method and the Jacobi pseudospectral method for three dimensional singular problem. Let

$$\begin{aligned}
&\Gamma_1=\{\,(-1,x_2,x_3)\mid -1\le x_2,x_3\le 1\,\}, &&\Gamma_2=\{\,(x_1,-1,x_3)\mid -1\le x_1,x_3\le 1\,\},\\
&\Gamma_3=\{\,(x_1,x_2,-1)\mid -1\le x_1,x_1\le 1\,\}, &&\Gamma_4=\{\,(1,x_2,x_3)\mid -1\le x_2,x_3\le 1\,\},\\
&\Gamma_5=\{\,(x_1,1,x_3)\mid -1\le x_1,x_3\le 1\,\}, &&\Gamma_6=\{\,(x_1,x_2,1)\mid -1\le x_1,x_1\le 1\,\}.
\end{aligned}$$

We consider the following model problem

$$\begin{aligned}
&-\partial_{x_1}(a_1(x)\partial_{x_1}U(x))-\partial_{x_2}(a_2(x)\partial_{x_2}U(x))-\partial_{x_3}(a_3(x)\partial_{x_3}U(x))+a_0(x)U(x)\\
&=f(x),\quad x\in\Omega. \qquad (5.1)
\end{aligned}$$

Without losing generality, we suppose that

$$\begin{aligned}
&a_1(x)=\tilde{a}_1(x)\chi_1^{(\alpha,\beta)}(x), &&a_2(x)=\tilde{a}_2(x)\chi_2^{(\alpha,\beta)}(x),\\
&a_3(x)=\tilde{a}_3(x)\chi_3^{(\alpha,\beta)}(x), &&a_0(x)=\tilde{a}_0(x)\chi^{(\gamma,\delta)}(x),
\end{aligned} \qquad (5.2)$$

where

$$\begin{aligned} &\tilde{a}_q(x)\in F^{s_q,\eta}, \ \ s_q=(\hat{s}_q,\bar{s}_q,\tilde{s}_q), \ \ \hat{s}_q,\bar{s}_q,\tilde{s}_q\geq 1, \ \ 0\leq\eta\leq\tfrac{1}{2}, \ q=1,2,3,\\ &\tilde{a}_q(x)\geq(\tilde{a}_q)_{min}>0, \ x\in\bar{\Omega}, \ q=1,2,3. \end{aligned} \tag{5.3}$$

We look for singular solution of (5.1) such that

$$\lim_{x\to\Gamma_1\cup\Gamma_4} a_1(x)\partial_{x_1}U(x)=0, \quad \lim_{x\to\Gamma_2\cup\Gamma_5} a_2(x)\partial_{x_2}U(x)=0, \quad \lim_{x\to\Gamma_3\cup\Gamma_6} a_3(x)\partial_{x_3}U(x)=0.$$

Let

$$\begin{aligned} A_{\alpha,\beta,\gamma,\delta}(u,v)=&(\tilde{a}_1(x)\partial_{x_1}u,\partial_{x_1}v)_{\chi_1^{(\alpha,\beta)}}+(\tilde{a}_2(x)\partial_{x_2}u,\partial_{x_2}v)_{\chi_2^{(\alpha,\beta)}}\\ &+(\tilde{a}_3(x)\partial_{x_3}u,\partial_{x_3}v)_{\chi_3^{(\alpha,\beta)}}+(\tilde{a}_0(x)u,v)_{\chi^{(\gamma,\delta)}}, \quad \forall\, u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega). \end{aligned}$$

A weak formulation of (5.1) is to find $U\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ such that

$$A_{\alpha,\beta,\gamma,\delta}(U,v)=(f,v)_{L^2(\Omega)}, \ \forall v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega). \tag{5.4}$$

It can be verified that for any $u,v\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$,

$$|A_{\alpha,\beta,\gamma,\delta}(u,v)|\leq\left(\sum_{q=0}^{3}\|\tilde{a}_q(x)\|_{L^\infty(\Omega)}\right)\|u\|_{1,\alpha,\beta,\gamma,\delta}\|v\|_{1,\alpha,\beta,\gamma,\delta}, \tag{5.5}$$

$$|A_{\alpha,\beta,\gamma,\delta}(u,u)|\geq \min_{0\leq q\leq 3}((\tilde{a}_q)_{min})\|u\|^2_{1,\alpha,\beta,\gamma,\delta}. \tag{5.6}$$

Therefore, if $f\in L^2_{\chi^{(-\gamma,-\delta)}}(\Omega)$, then by (5.5) and (5.6) and Lax-milgram lemma, then (5.4) has a unique solution such that $\|U\|_{1,\alpha,\beta,\gamma,\delta}\leq c\|f\|_{\chi^{(-\gamma,-\delta)}}$.

Now, let $u_N\in\mathcal{P}_N(\Omega)$ be the approximation of U, satisfying

$$A_{\alpha,\beta,\gamma,\delta}(u_N,\phi)=(f,\phi)_{L^2(\Omega)}, \qquad \forall\phi\in\mathcal{P}_N(\Omega). \tag{5.7}$$

Due to (5.5) and (5.6), the solution of equation (5.7) is unique and $\|u_N\|_{1,\alpha,\beta,\gamma,\delta}\leq c\|f\|_{\chi^{(-\gamma,-\delta)}}$.

According to (5.4) and (5.7), we have that

$$A_{\alpha,\beta,\gamma,\delta}(u_N-U,\phi)=0, \qquad \forall\phi\in\mathcal{P}_N(\Omega). \tag{5.8}$$

Theorem 5.1. *If* (3.1) *and* (5.2) *hold, and* $B_{r,\alpha,\beta,\gamma,\delta}(U)$ *is finite , then for* $1\leq r_i\leq N_i+1$, $i=1,2,3$

$$\|u_N-U\|_{1,\alpha,\beta,\gamma,\delta}\leq c(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3})B_{r,\alpha,\beta,\gamma,\delta}(U),$$

where the constant c depends only on the norms $\|\tilde{a}_i\|_{C(\Omega)}$, $i=0,1,2,3$.

Proof. Let $U_N = P^1_{N,\alpha,\beta,\gamma,\delta,\Omega}U$. By using (5.6), (5.8) and Theorem 3.2 successively, we get that

$$\begin{aligned}&\min_{0\le q\le 3}((\tilde a_q)_{min})\|u_N-U_N\|^2_{1,\alpha,\beta,\gamma,\delta}\\ \le& c\left(\sum_{q=0}^{3}\|\tilde a_q(x)\|_{C(\Omega)}\right)\left(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\right)B_{r,\alpha,\beta,\gamma,\delta}(U)\|u_N-U_N\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|u_N-U_N\|_{1,\alpha,\beta,\gamma,\delta}\\ \le& c\frac{\sum\limits_{q=0}^{3}\|\tilde a_q(x)\|_{C(\Omega)}}{\min\limits_{0\le q\le 3}((\tilde a_q)_{min})}\left(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3}\right)B_{r,\alpha,\beta,\gamma,\delta}(U).\end{aligned}\tag{5.9}$$

By using Theorem 3.2 and (5.9), we derive that

$$\|u_N-U\|_{1,\alpha,\beta,\gamma,\delta}\le c(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3})B_{r,\alpha,\beta,\gamma,\delta}(U),$$

where c depends on $(\tilde a_q)_{min}$ and $\|\tilde a_q(x)\|_{C(\Omega)}$, $q=0,1,2,3$. □

Next, we consider the pseudospectral method for problem (5.1). Let $\hat a_q = I_{\mathbb{L},N/2,0,0}\tilde a_q$, $q=0,1,2,3$ and $\hat f(x)=\chi^{(-\gamma,-\delta)}f(x)$. Assuming $\hat f(x)\in C(\Omega)$, we let

$$\begin{aligned}\hat A_{\alpha,\beta,\gamma,\delta,N}(u,v)=&(\hat a_1(x)\partial_{x_1}u,\partial_{x_1}v)_{\chi_1^{(\alpha,\beta)},\mathbb{G},N}+(\hat a_2(x)\partial_{x_2}u,\partial_{x_2}v)_{\chi_2^{(\alpha,\beta)},\mathbb{G},N}\\ &+(\hat a_3(x)\partial_{x_3}u,\partial_{x_3}v)_{\chi_3^{(\alpha,\beta)},\mathbb{G},N}+(\hat a_0(x)u,v)_{\chi^{(\gamma,\delta)},\mathbb{G},N}.\end{aligned}$$

The Jacobi pseudospectral scheme of (5.1) is to find $u_N\in\mathcal{P}_N(\Omega)$ such that

$$\hat A_{\alpha,\beta,\gamma,\delta,N}(u_N,\phi)=(\hat f,\phi)_{\chi^{(\gamma,\delta)},\mathbb{G},N},\qquad \forall\phi\in\mathcal{P}_N(\Omega).\tag{5.10}$$

By (4.3), if $\phi,\psi\in\mathcal{P}_N(\Omega)$, we have that

$$\begin{aligned}&|\hat A_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)|\\ \le&|(\hat a_1(x)\partial_{x_1}\phi,\partial_{x_1}\psi)_{\chi_1^{(\alpha,\beta)},\mathbb{G},N}|+|(\hat a_2(x)\partial_{x_2}\phi,\partial_{x_2}\psi)_{\chi_2^{(\alpha,\beta)},\mathbb{G},N}|\\ &+|(\hat a_3(x)\partial_{x_3}\phi,\partial_{x_3}\psi)_{\chi_3^{(\alpha,\beta)},\mathbb{G},N}|+|(\hat a_0(x)\phi,\psi)_{\chi^{(\gamma,\delta)},\mathbb{G},N}|\\ \le&\Big(\|\hat a_1\|_{C(\Omega)}+\|\hat a_2\|_{C(\Omega)}+\|\hat a_3\|_{C(\Omega)}+\|\hat a_0\|_{C(\Omega)}\Big)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}\tag{5.11}$$

Thanks to (4.27) and (5.3), there exists a constant $c^*>0$, depending only on $\|\tilde a_q\|_{F^{s_q,1}}$, $q=0,1,2,3$, such that

$$\sum_{q=0}^{3}\|\hat a_q\|_{C(\Omega)}\le c^*.\tag{5.12}$$

Therefore, for any $\phi,\psi\in\mathcal{P}_N(\Omega)$

$$|\hat{A}_{\alpha,\beta,\gamma,\delta,N}(\phi,\psi)|\le c^*\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\|\psi\|_{1,\alpha,\beta,\gamma,\delta}. \tag{5.13}$$

Besides, for any $\phi\in\mathcal{P}_N(\Omega)$ and suitably large N, according to (4.3), (4.27) and (5.3), the coefficient $\hat{a}_q(x)$, $q=0,1,2,3$, are uniformly bounded below by a constant $c_*>0$,

$$|\hat{A}_{\alpha,\beta,\gamma,\delta,N}(\phi,\phi)|\ge c_*\|\phi\|^2_{1,\alpha,\beta,\gamma,\delta}. \tag{5.14}$$

Hence, by the Lax-Milgram lemma, (5.9) has a unique solution such that $\|u_N\|_{1,\alpha,\beta,\gamma,\delta}\le c\|I_{G,N,\gamma,\delta}\tilde{f}\|_{\chi^{(\gamma,\delta)}}$.

Theorem 5.2. *Let* (5.2) *and* (5.3) *hold, and* $\gamma_q\le\tilde{\alpha}_q$, $\gamma_q\le\bar{\alpha}_q$, $\delta_q\le\tilde{\beta}_q$, $\delta_q\le\bar{\beta}_q$, $q=1,2,3$. *If* $B_{r,\alpha,\beta,\gamma,\delta}(U)$ *and* $D_{r,\gamma,\delta}(\hat{f})$ *are finite, then for* $1\le r_i\le N_i+1$, $i=1,2,3$ *and* $2\le r_i'\le N_i+1$, $i=1,2,3$,

$$\begin{aligned}&\|u_N-U\|_{1,\alpha,\beta,\gamma,\delta}\\ \le& c\Big(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3})B_{r,\alpha,\beta,\gamma,\delta}(U)+d(s,N,\tilde{a},U)\\ &+\big(N_1^{-r_1'}+N_2^{-r_2'}+N_3^{-r_3'}+N_1^{-1}N_2^{1-r_2'}+N_1^{-1}N_3^{1-r_3'}+N_2^{-1}N_3^{1-r_3'}\\ &+N_1^{-1}N_2^{-1}N_3^{2-r_3'}\Big)D_{r,\gamma,\delta}(\hat{f}),\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}&d(s,N,\tilde{a},U)\\ =&c\Big((\frac{N_1}{2})^{\frac12-\hat{s}_0}+(\frac{N_2}{2})^{\frac12-\bar{s}_0}+(\frac{N_3}{2})^{\frac12-\tilde{s}_0}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{\frac32-\bar{s}_0}+(\frac{N_1}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_0}\\ &+(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_0}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_0}\Big)|\tilde{a}_0|_{F^{s_0,\eta}(\Omega)}\|U\|_{\chi^{(\gamma,\delta)}}\\ &+c\Big((\frac{N_1}{2})^{\frac12-\hat{s}_1}+(\frac{N_2}{2})^{\frac12-\bar{s}_1}+(\frac{N_3}{2})^{\frac12-\tilde{s}_1}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{\frac32-\bar{s}_1}+(\frac{N_1}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_1}\\ &+(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_1}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_1}\Big)|\tilde{a}_1|_{F^{s_1,\eta}(\Omega)}\|\partial_{x_1}U\|_{\chi_1^{(\alpha,\beta)}}\\ &+c\Big((\frac{N_1}{2})^{\frac12-\hat{s}_2}+(\frac{N_2}{2})^{\frac12-\bar{s}_2}+(\frac{N_3}{2})^{\frac12-\tilde{s}_2}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{\frac32-\bar{s}_2}+(\frac{N_1}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_2}\\ &+(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_2}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_2}\Big)|\tilde{a}_2|_{F^{s_2,\eta}(\Omega)}\|\partial_{x_2}U\|_{\chi_2^{(\alpha,\beta)}}\\ &+c\Big((\frac{N_1}{2})^{\frac12-\hat{s}_3}+(\frac{N_2}{2})^{\frac12-\bar{s}_3}+(\frac{N_3}{2})^{\frac12-\tilde{s}_3}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{\frac32-\bar{s}_3}+(\frac{N_1}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_3}\\ &+(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_3}+(\frac{N_1}{2})^{-\frac12}(\frac{N_2}{2})^{-\frac12}(\frac{N_3}{2})^{\frac32-\tilde{s}_3}\Big)|\tilde{a}_3|_{F^{s_3,\eta}(\Omega)}\|\partial_{x_3}U\|_{\chi_3^{(\alpha,\beta)}}.\end{aligned}$$

Proof. Let $U_N=P^1_{N,\alpha,\beta,\gamma,\delta}U$. By (5.14), (5.10), (5.4) and the definition of $\hat{f}$, we have that

$$\begin{aligned}c_*\|u_N-U_N\|^2_{1,\alpha,\beta,\gamma,\delta}\le&|(\hat{f},u_N-U_N)_{\chi^{(\gamma,\delta)},G,N}-\hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,u_N-U_N)\\&+A_{\alpha,\beta,\gamma,\delta}(U,u_N-U_N)-(\hat{f},u_N-U_N)_{\chi^{(\gamma,\delta)}}|.\end{aligned}$$

Thus

$$\begin{aligned}\|u_N-U_N\|^2_{1,\alpha,\beta,\gamma,\delta}\le& c\Big(|A_{\alpha,\beta,\gamma,\delta}(U,u_N-U_N)-\hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,u_N-U_N)|\\&+|(\hat{f},u_N-U_N)_{\chi^{(\gamma,\delta)},G,N}-(\hat{f},u_N-U_N)_{\chi^{(\gamma,\delta)}}|\Big).\end{aligned}\tag{5.16}$$

For simplicity, let

$$\begin{aligned}\hat{A}_{\alpha,\beta,\gamma,\delta}(u,\phi)=&(\hat{a}_1(x)\partial_{x_1}u,\partial_{x_1}\phi)_{\chi_1^{(\alpha,\beta)}}+(\hat{a}_2(x)\partial_{x_2}u,\partial_{x_2}\phi)_{\chi_2^{(\alpha,\beta)}}+(\hat{a}_3(x)\partial_{x_3}u,\partial_{x_3}\phi)_{\chi_3^{(\alpha,\beta)}}\\&+(\hat{a}_0(x)u,\phi)_{\chi^{(\gamma,\delta)}},\quad \forall u\in H^1_{\alpha,\beta,\gamma,\delta}(\Omega),\quad \forall\phi\in\mathcal{P}_N(\Omega).\end{aligned}$$

Then

$$\begin{aligned}&A_{\alpha,\beta,\gamma,\delta}(U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)\\=&A_{\alpha,\beta,\gamma,\delta}(U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi)+\hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi).\end{aligned}\tag{5.17}$$

Using (4.27) for $\|\hat{a}_q-\tilde{a}_q\|_{C(\Omega)},q=0,1,2,3$, we obtain that

$$\begin{aligned}&|A_{\alpha,\beta,\gamma,\delta}(U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi)|\\\le&(\|\hat{a}_0-\tilde{a}_0\|_{C(\Omega)}\|U\|_{\chi^{(\gamma,\delta)}}+\|\hat{a}_1-\tilde{a}_1\|_{C(\Omega)}\|\partial_{x_1}U\|_{\chi_1^{(\alpha,\beta)}}\\&+\|\hat{a}_2-\tilde{a}_2\|_{C(\Omega)}\|\partial_{x_2}U\|_{\chi_2^{(\alpha,\beta)}}+\|\hat{a}_3-\tilde{a}_3\|_{C(\Omega)}\|\partial_{x_3}U\|_{\chi_3^{(\alpha,\beta)}})\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\\\le& d(s,N,\tilde{a},U)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}\tag{5.18}$$

By the definitions of $\hat{A}_{\alpha,\beta,\gamma,\delta,N}(u,\phi)$ and $\hat{a}_q(x)$, we get

$$\hat{A}_{\alpha,\beta,\gamma,\delta,N}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U,\phi)=\hat{A}_{\alpha,\beta,\gamma,\delta}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U,\phi).$$

Thanks to the above equality, (5.5), (5.13) and (3.3), we have that

$$\begin{aligned}&|\hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)|\\\le&|\hat{A}_{\alpha,\beta,\gamma,\delta}(U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U,\phi)+\hat{A}_{\alpha,\beta,\gamma,\delta}(P^1_{N/2,\alpha,\beta,\gamma,\delta}U,\phi)-\hat{A}_{\alpha,\beta,\gamma,\delta,N}(U_N,\phi)|\\\le& c^*(\|P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U\|_{1,\alpha,\beta,\gamma,\delta}+\|P^1_{N/2,\alpha,\beta,\gamma,\delta}U-U_N\|_{1,\alpha,\beta,\gamma,\delta})\|\phi\|_{1,\alpha,\beta,\gamma,\delta}\\\le& cc^*(N_1^{1-r_1}+N_2^{1-r_2}+N_3^{1-r_3})B_{r,\alpha,\beta,\gamma,\delta}(U)\|\phi\|_{1,\alpha,\beta,\gamma,\delta}.\end{aligned}\tag{5.19}$$

Moreover, the definition of $\hat{f}(x)$, (4.3) and (4.11) yield that

$$\begin{aligned}|(f,\phi)-(\hat{f},\phi)_{\chi^{(\gamma,\delta)},G,N}| \le c\Big(&N_1^{-r_1'}+N_2^{-r_2'}+N_3^{-r_3'}+N_1^{-1}N_2^{1-r_2'}+N_1^{-1}N_3^{1-r_3'}\\ &+N_2^{-1}N_3^{1-r_3'}+N_1^{-1}N_2^{-1}N_3^{2-r_3'}\Big)D_{r,\gamma,\delta}(\hat{f})\|\phi\|_{1,\alpha,\beta\gamma,\delta}.\end{aligned} \tag{5.20}$$

A combination of (5.16)-(5.20) with $\phi=u_N-U_N$, and (3.3) leads to the desired result (5.15). □

6 Numerical results

In this section, we present some numerical results to demonstrate the efficiency of the proposed methods. We consider the singular problem (5.1) with

$$\begin{aligned}&a_1(x)=(1-x_1^2)^2(1-x_2^2)(1-x_3^2), \qquad a_2(x)=(1-x_1^2)(1-x_2^2)^2(1-x_3^2),\\ &a_3(x)=(1-x_1^2)(1-x_2^2)(1-x_3^2)^2, \qquad a_0(x)=(1-x_1^2)(1-x_2^2)(1-x_3^2).\end{aligned}$$

Take the test function

$$U(x)=\arcsin(x_1x_2x_3)e^{x_1x_2x_3}. \tag{6.1}$$

Clearly, $|\partial_{x_q}U|\to\infty, q=1,2,3$, as x tends to $\Gamma_j\cup\Gamma_{j+3}$, $1\le j\le 3$.

Let ζ_{G,N_q,j_q} and ω_{G,N_q,j_q} be nodes and weights of the one-dimensional Legendre-Gauss quadrature. We measure the error of the spectral method by

$$E(u_N)=\left(\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}\sum_{j_3=0}^{N_3}(u_N(\zeta_{G,N_1,j_1},\zeta_{G,N_2,j_2},\zeta_{G,N_3,j_3})-U(\zeta_{G,N_1,j_1},\zeta_{G,N_2,j_2},\zeta_{G,N_3,j_3}))^2 \times\omega_{G,N_1,j_1}\omega_{G,N_2,j_2}\omega_{G,N_3,j_3}\right)^{\frac{1}{2}},$$

and the error of the pseudospectral method errors by

$$E(Psu_N)=\left(\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}\sum_{j_3=0}^{N_3}\Big(Ps\ u_N(\zeta_{G,N_1,j_1},\zeta_{G,N_2,j_2},\zeta_{G,N_3,j_3})-U(\zeta_{G,N_1,j_1},\zeta_{G,N_2,j_2},\zeta_{G,N_3,j_3}))^2 \times\omega_{G,N_1,j_1}\omega_{G,N_2,j_2}\omega_{G,N_3,j_3}\right)^{\frac{1}{2}}.$$

In table 1, we present the errors $E(u_N)$ and $E(Psu_N)$ with $N_1=N_2=N_3=N$. Obviously, the two schemes provide accurate numerical solutions, and the numerical solutions converge to the exact solution rapidly as N increases. This confirms the theoretical analysis well.

Table 1: Errors of spectral method and pseudospectral method

N	8	16	24	32	40	48
$E(u_N)$	5.2596e-004	5.9994e-005	1.4615e-005	5.1478e-006	2.2508e-006	6.4626e-007
$E(Psu_N)$	5.3144e-004	6.2570e-005	1.6229e-005	6.1200e-006	2.8487e-006	1.5156e-006
N	56	64	72	80	88	96
$E(u_N)$	3.1346e-007	9.6804e-008	9.0445e-008	5.2239e-008	3.1245e-008	1.2128e-008
$E(Psu_N)$	8.8434e-007	5.5221e-007	3.6324e-007	2.4903e-007	1.7657e-007	1.5069e-007

In Figs. 1-3, we compare the exact solution and the numerical solution of (5.7) and (5.9) with $N=32$. Clearly, the numerical solutions of spectral scheme and pseudospectral scheme match the exact solution very well when $z=0.9$, 0, -0.9, respectively.

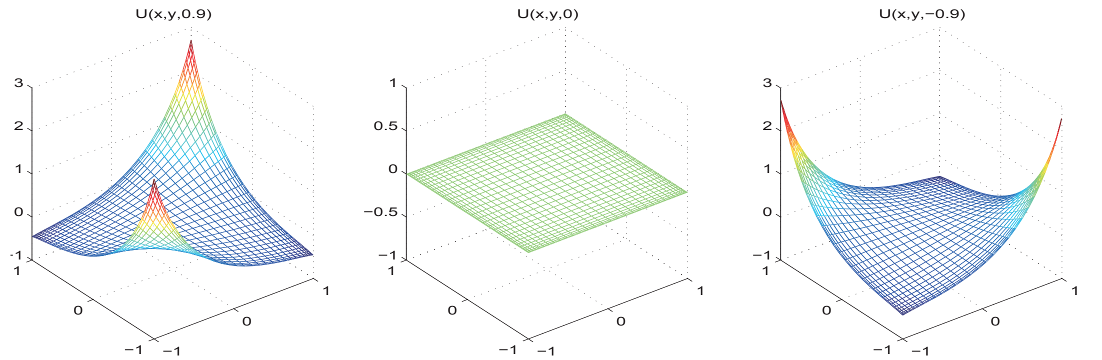

Figure 1: Exact solutions

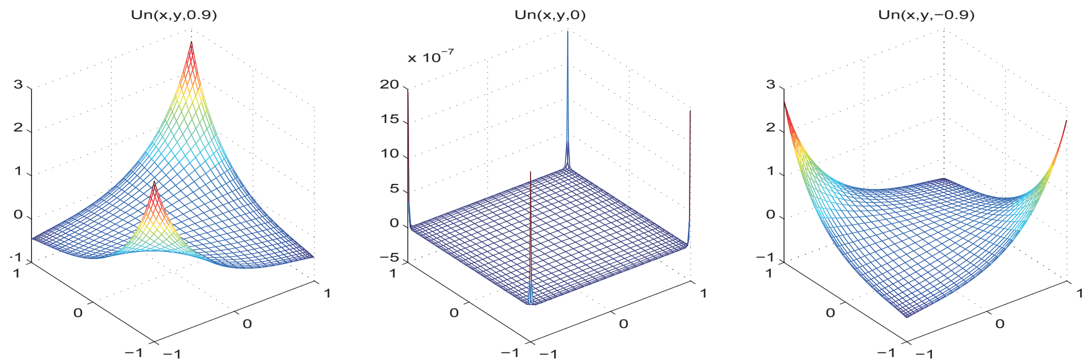

Figure 2: Solutions of spectral scheme

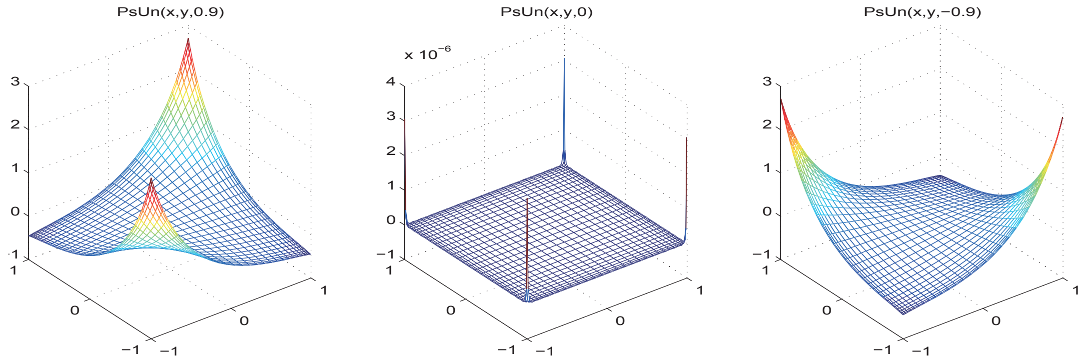

Figure 3: Solutions of pseudospectral scheme

7 Concluding Remarks

In this paper, we establish some basic results on the Non-isotropic 3-dimensional Jacobi spectral approximation and Jacobi-Gauss type interpolation, with which the convergence of proposed schemes are proved. These results play important role in designing and analyzing Jacobi spectral methods and pseudospectral methods for various practical problems. As examples, we consider a singular problem in three dimensions. The numerical results demonstrate the spectral accuracy of proposed schemes, and agreed well with the theoretical analysis.

We also derive a series of sharp results on the Legendre-Gauss type interpolation and the related Bernstein-Jackson type inequalities in three dimensions, which improve and generalize the existing results. They are very useful for pseudospectral method of partial differential equations with non-constant coefficients, as well as numerical solutions of initial value problems of nonlinear ordinary differential equations.

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