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A NEW FINITE ELEMENT APPROXIMATION OF A STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM*

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Abstract

In this paper, we study numerical methods for an optimal control problem with pointwise state constraints. The traditional approaches often need to deal with the delta-singularity in the dual equation, which causes many difficulties in its theoretical analysis and numerical approximation. In our new approach we reformulate the state-constrained optimal control as a constrained minimization problems only involving the state, whose optimality condition is characterized by a fourth order elliptic variational inequality. Then direct numerical algorithms (nonconforming finite element approximation) are proposed for the inequality, and error estimates of the finite element approximation are derived. Numerical experiments illustrate the effectiveness of the new approach.

Mathematics subject classification: 49J20, 65N30.
Key words: Optimal control problem, State-constraints, Fourth order variational inequalities, Nonconforming finite element method.

1. Introduction

In this paper, we consider the following state-constrained optimal control problem:

$$\min_{y\le\varphi}\left\{\frac{1}{2}\int_\Omega (y-y_0)^2dx+\frac{\alpha}{2}\int_\Omega u^2dx\right\} \tag{1.1}$$

subject to

$$\begin{aligned} -\Delta y = u \quad &\text{in} \quad \Omega \\ y = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

where $\alpha > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^2$ with the Lipschitz continuous boundary $\partial\Omega$, $y_0 \in L^2(\Omega)$ is the desired state, and φ is a given function. We further assume that $\varphi|_{\partial\Omega} > 0$, and more details will be specified later.

Such a state-constrained optimal control is a very important model in many applications and there has already existed much research on the numerical approximation of the above state constrained optimal control problem in the literature. Many numerical strategies were proposed and both a priori and a posteriori error analysis were investigated. At first, we should mention

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the work of Casas in [7], where the optimality conditions and important theoretical analysis of the problem were provided. For the standard finite element approximation of the control problem, a priori error estimates were derived by Deckelnick and Hinze in [11], where non-classic techniques were developed to handle the delta-singularity of the co-stated equation, see below. An augmented Lagrangian method was proposed to solve state and control constrained optimal control problems by Bergounioux and Kunisch in [3]. They also proposed another method: a primal-dual strategy to solve problem (1.1) in [4]. Casas proved convergence of finite element approximations to optimal control problems for semi-linear elliptic equations with finitely many state constraints in [8]. Casas and Mateos extended these results in [9] to a less regular setting for the states, and proved convergence of finite element approximations to semi-linear distributed and boundary control problems. In [25], the state-constrained control problem was approximated by a sequence of control-constrained control problems, and then the interior point method was applied to approximating the solutions. In recent years, a level set approach was applied to state-constrained problems in [16].

Furthermore all the research mentioned above was based on the first order optimality conditions of the control problem, in which an adjoint state p and a Lagrange multiplier λ are introduced. The first order optimality conditions can be stated as: The pari $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ is the unique solution of (1.1) if and only if there exist an adjoint state $p \in L^2(\Omega)$ and a Lagrange multiplier $\lambda \in \mathcal{M}(\Omega)$ such that

$$\begin{cases} -\Delta y = u \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega \\ -(p, \Delta w)_\Omega + < \lambda, w >_{\mathcal{M},\mathcal{C}} = (y_0 - y, w)_\Omega, \quad \forall w \in H_0^1(\Omega) \cap H^2(\Omega), \\ < \lambda, z - y >_{\mathcal{M},\mathcal{C}} \le 0, \quad \forall z \in \mathcal{C}(\Omega), \quad z \le \varphi, \\ p = \alpha u, \\ y \le \varphi \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

where $(\cdot,\cdot)_\Omega$ denotes the L^2 inner product in Ω, $< \cdot, \cdot >_{\mathcal{M},\mathcal{C}}$ denotes the duality pairing between $\mathcal{C}(\Omega)$ and $\mathcal{M}(\Omega)$. We denote by $\mathcal{M}(\Omega)$ the space of real regular Borel measures on Ω and recall that it can be identified with the dual $\mathcal{C}^*(\Omega)$ of $\mathcal{C}(\Omega)$. In particular, every element $\lambda \in \mathcal{C}^*(\Omega)$ generates an element $[\lambda] \in \mathcal{M}(\Omega)$ such that $< \lambda, y >_{\mathcal{C}^*,\mathcal{C}} = \int_\Omega y \; d[\lambda]$ for all $y \in \mathcal{C}(\Omega)$. The details can be found in [4, 5, 16], for example.

One of the main computational difficulties in solving the above system is that the multiplier λ is often a delta measure, which has infinite values at some unknown points on the free boundary of the coincidence set $\{x : y = \varphi\}$. Whatever discretization methods are used, special care needs to be taken for these (unknown) areas in order to obtain reasonable computational efficiency. In finite element method, normally adaptive meshes are needed so that they are refined around these points guided by some error estimators. This is the main motivation of a posteriori error analysis of the finite element method. In this regard, a goal-oriented adaptive finite element concept was developed in [14], while Hoppe and Kieweg provided a posteriori error estimators of residual-type for the state constrained optimal control problem in [17]. However there seemed to still exist many issues in the formulation and analysis of these a posteriori error estimators due to the presence of the delta measure.

In this paper, we adopt a different approach for approximating this state-constrained optimal control problem, which avoids using the first order optimality conditions. The main idea is: substitute the control u in the minimizing functional (1.1) by $u = -\Delta y$, which is based on the state equation, and reformulate it as a constrained minimization problem involving only state

y. So the original problem (1.1) can be restated as follows

$$\min_{y\le\varphi}\left\{\frac{1}{2}\int_\Omega (y-y_0)^2dx+\frac{\alpha}{2}\int_\Omega |\Delta y|^2dx\right\}. \tag{1.3}$$

It will be shown in the next sections that the minimizing problem (1.3) is equivalent to a fourth order variational inequality. Thus we convert the state-constrained optimal control problem to a variational inequality problem. Consequently our approach only needs to solve a variational inequality of fourth order to obtain the state y, which is in fact quite smooth. Then the numerical procedures and the theoretical analysis seem to become simpler, and in fact often were well-studied already in the literature. Furthermore this idea seems to be applicable for wider range of state-constrained control problems.

The paper is organized as follows: In Section 2, we introduce a new weak formulation for the state constrained optimal control problem. Then its finite element approximation is proposed in Section 3. In Section 4 we discuss regularity of the the solution for the state-constrained optimal control problem and then derive a priori error estimates. In Section 5, we discuss the numerical methods for solving the fourth order variational inequality, and present some numerical examples to illustrate effectiveness of our new approach.

2. The New Formulation

Let Ω be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $||\cdot||_{m,q,\Omega}$ and seminorm $|\cdot|_{m,q,\Omega}$. We set $W_0^{1,q}(\Omega)\equiv\{w\in W^{1,q}(\Omega):\ w|_{\partial\Omega}=0\}$. We denote $W^{m,2}(\Omega)\big(W_0^{m,2}(\Omega)\big)$ by $H^m(\Omega)\big(H_0^m(\Omega)\big)$. In addition, c or C denotes a general positive constant independent of h.

Let

$$\begin{aligned} a(u,v) &= \int_\Omega (\alpha\Delta u\cdot\Delta v+u\cdot v)dx \quad \forall u,v\in H_0^1(\Omega)\cap H^2(\Omega),\\ (f,v) &= \int_\Omega f\cdot v dx \quad \forall f,v\in L^2(\Omega). \end{aligned}$$

Let K be a close convex set defined by

$$K=\{v:\ v\in H_0^1(\Omega)\cap H^2(\Omega),\ \ v\le\varphi \quad \text{a.e. in } \Omega\}, \tag{2.1}$$

where φ is a given function. Then it is easy to prove that the minimizing problem (1.3) is equivalent to the following variational inequality

$$\begin{cases} \text{Find } y\in K, \text{ such that} \\ a(y,w-y)\ge(y_0,w-y),\ \ \forall w\in K. \end{cases} \tag{2.2}$$

Let $V=H_0^1(\Omega)\cap H^2(\Omega)$. It is clear that the bilinear form $a(y,w)$ is continuous over $V\times V$ and $V-$elliptic. Since K is a closed, convex and non-empty subset of V, problem (2.2) has a unique solution by the standard argument. Moreover, with the assumption $\varphi|_{\partial\Omega}>0$, it can be proven that for the problem (1.1), the optimal control $u|_{\partial\Omega}=0$, which means $\Delta y|_{\partial\Omega}=0$ (see [16] for more details). We further assume that $\varphi\in H^2(\Omega)$ in our theoretical analysis (see Theorem 4.3 and Remark 4.5).

We define the non-coincidence set and coincidence set with respect to the state y by

$$\Omega^+ = \{x \in \Omega : \; y(x) < \varphi(x)\}, \tag{2.3}$$

$$\Omega^0 = \{x \in \Omega : \; y(x) = \varphi(x)\}. \tag{2.4}$$

Then we can derive a boundary value problem satisfied by (2.2).

Lemma 2.1. *The fourth order variational inequality* (2.2) *is equivalent to the following boundary value problems:*

$$\begin{cases} \alpha\Delta^2 y + y = y_0 & \text{in} \quad \Omega^+, \\ \alpha\Delta^2 y + y \le y_0 & \text{in} \quad \Omega^0, \\ y \le \varphi & \text{in} \quad \Omega, \\ y = \Delta y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5}$$

Proof. Using the Green's formula in (2.2), we have that for all $w \in K$,

$$\int_\Omega (\alpha\Delta^2 y + y - y_0)\cdot(w-y)dx + \int_{\partial\Omega} \alpha\Delta y \frac{\partial(w-y)}{\partial n} ds - \int_{\partial\Omega} \alpha \frac{\partial(\Delta y)}{\partial n}(w-y)ds \ge 0. \tag{2.6}$$

Since $y, w \in K$, we have

$$w|_{\partial\Omega} = y|_{\partial\Omega} = 0, \quad \text{and} \quad y \le \varphi \;\text{ in }\; \Omega. \tag{2.7}$$

Thus (2.6) reduces to

$$\int_\Omega (\alpha\Delta^2 y + y - y_0)\cdot(w-y)dx + \int_{\partial\Omega} \alpha\Delta y \frac{\partial(w-y)}{\partial n} ds \ge 0, \;\; \forall w \in K. \tag{2.8}$$

Let $w = y + \psi \in K$, $\forall \psi \in \mathcal{D}(\Omega) : \; \psi \le 0$. We have

$$\int_\Omega (\alpha\Delta^2 y + y - y_0)\cdot\psi dx \ge 0, \quad \forall \psi \in \mathcal{D}(\Omega) : \; \psi \le 0,$$

which implies that

$$\alpha\Delta^2 y + y \le y_0 \quad \text{in} \quad \Omega. \tag{2.9}$$

Note that $\forall \theta(x) \in \mathcal{D}(\Omega)$, $0 \le \theta(x) \le 1$, we have $w = \theta(x)\cdot\varphi + \big(1-\theta(x)\big)\cdot y \in K$. Considering (2.8), we have

$$\int_\Omega (\alpha\Delta^2 y + y - y_0)\,\theta(x)\cdot(\varphi - y)dx \ge 0, \quad \forall \theta \in \mathcal{D}(\Omega) : \; 0 \le \theta(x) \le 1;$$

that is

$$\int_{\Omega^+} (\alpha\Delta^2 y + y - y_0)\,\theta(x)\cdot(\varphi - y)dx \ge 0, \quad \forall \theta \in \mathcal{D}(\Omega) : \; 0 \le \theta(x) \le 1, \tag{2.10}$$

which implies

$$\alpha\Delta^2 y + y - y_0 \ge 0 \;\text{ in }\; \Omega^+. \tag{2.11}$$

Therefore, it follows from (2.9) and (2.11) that

$$\alpha\Delta^2 y + y - y_0 = 0 \ \text{ in } \Omega^+. \tag{2.12}$$

Moreover, note that $y|_{\partial\Omega} = 0 < \varphi|_{\partial\Omega}$. Then $\forall z \in \partial\Omega$, there exists a subdomain ω_z of z with $\mathrm{dist}(z, \partial\omega_z \setminus \partial\Omega) = r_z > 0$ and $y|_{\omega_z} < \varphi$. Therefore for all $\theta \in H_0^1(\omega_z) \cap H^2(\omega_z)$ there exists a positive number ϵ such that $y \pm \epsilon\theta \leq \varphi$, and hence $w = y \pm \epsilon\theta \in K$. Then it follows from (2.8) and (2.12) that

$$\pm\epsilon\alpha \int_{\partial\Omega} \Delta y \frac{\partial\theta}{\partial n} ds \geq 0, \quad \forall\theta \in H_0^1(\omega_z) \cap H^2(\omega_z), \tag{2.13}$$

which implies

$$\Delta y|_{\partial\Omega} = 0. \tag{2.14}$$

Then (2.5) follows from (2.7), (2.9), (2.12) and (2.14). □

It should be pointed that the method proposed in this paper can be extended to more general cases. Consider the problem:

$$\min \left\{ \frac{1}{2} \int_\Omega (y - y_0)^2 dx + \frac{\alpha}{2} \int_\Omega (u - u_0)^2 dx \right\} \tag{2.15}$$

subject to

$$\begin{aligned} & -\Delta y = u \ \text{ in } \ \Omega \\ & y = 0 \ \text{ on } \ \partial\Omega. \\ & y \in K_1 \ \text{ and } \ u \in K_2, \end{aligned}$$

where K_1, K_2 are closed convex subsets of $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively.

As in the beginning of Section 2, we can reformulate the problem (2.15) as the following minimization problem:

$$\min_{y \in K_3} \left\{ \frac{1}{2} \int_\Omega (y - y_0)^2 dx + \frac{\alpha}{2} \int_\Omega |\Delta y + u_0|^2 dx \right\}, \tag{2.16}$$

where

$$K_3 = \{ y \in H^2(\Omega) \cap H_0^1(\Omega) : \ y \in K_1, \quad -\Delta y \in K_2 \}.$$

Then the equivalent variational form reads

$$\begin{cases} \text{Find } y \in K_3, \text{ such that} \\ a(y, w - y) \geq (y_0, w - y) + b(u_0, v), \quad \forall w \in K_3, \end{cases} \tag{2.17}$$

where

$$b(w, v) = -\int_\Omega w \cdot \Delta v, \quad \forall w \in L^2(\Omega), \ \ v \in H^2(\Omega).$$

3. Nonconforming Finite Element Approximation

In order to introduce the nonconforming finite element approximation of problem (2.2), we rewrite the bilinear form $a(y,w)$ as

$$\tilde{a}(y,w)=\int_\Omega(\alpha\partial_{ij}y\cdot\partial_{ij}w+y\cdot w)dx,\quad \forall y,w\in H_0^1(\Omega)\cap H^2(\Omega),$$

where $\partial_{ij}y,\ \partial_{ij}w$ denotes

$$\frac{\partial^2 y}{\partial x_i\partial x_j},\ \frac{\partial^2 w}{\partial x_i\partial x_j},$$

respectively, the summation convention of repeated indices is used. It is clear that

$$a(y,w)=\tilde{a}(y,w)\text{ when }y,w\in H_0^1(\Omega)\cap H^2(\Omega).$$

The problem (2.2) is then restated as

$$\begin{cases}\text{Find } y\in K,\ \text{such that}\\ \tilde{a}(y,w-y)\ge(y_0,w-y),\quad \forall w\in K.\end{cases}\tag{3.1}$$

Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let $\mathcal{T}^h$ be a partitioning of Ω^h into disjoint regular triangle τ, such that $\bar{\Omega}^h=\bigcup_{\tau\in\mathcal{T}^h}\bar{\tau}$. For simplicity, we assume that Ω is a polygon or polyhedron such that $\Omega^h=\Omega$. We denote $\mathcal{N}(\Omega)$ and $\mathcal{E}(\Omega)$ the unions of all the interior vertices and internal edges of the triangulation $\mathcal{T}^h$, $\mathcal{N}(\partial\Omega)$ and $\mathcal{E}(\partial\Omega)$ the unions of all the vertices and edges on the boundary $\partial\Omega$, respectively. We approximate $V=H_0^1(\Omega)\cap H^2(\Omega)$ by using the nonconforming Morley's triangular finite element. The finite dimensional space V_h is defined by

$$\begin{aligned}V_h=&\Big\{v_h\in L^2(\Omega):\ v_h|_\tau\in P_2(\tau),\ v_h\text{ is continuous at each vertex }a\in\mathcal{N}(\Omega),\\ &\int_e\left[\frac{\partial v_h}{\partial n}\right]ds=0,\quad\forall e\in\mathcal{E}(\Omega),v_h(a)=0,\quad\forall a\in\mathcal{N}(\partial\Omega)\Big\},\end{aligned}\tag{3.2}$$

where $[\frac{\partial v_h}{\partial n}]|_e$ denotes the jump of $\frac{\partial v_h}{\partial n}$ across the edge e.

We construct an interpolation operator $\Pi_h:\ H_0^1(\Omega)\cap H^2(\Omega)\to V_h$ as follows

$$\begin{aligned}&\Pi_h v\in V_h, &&\forall v\in V,\\ &\Pi_h v(a)=v(a), &&\forall a\in\mathcal{N}(\Omega)\cup\mathcal{N}(\partial\Omega),\\ &\int_e\frac{\partial}{\partial n}\Pi_h v ds=\int_e\frac{\partial v}{\partial n}ds, &&\forall e\in\mathcal{E}(\Omega)\cup\mathcal{E}(\partial\Omega).\end{aligned}$$

Then it is natural to approximate the convex set K and the bilinear form $\tilde{a}(y,w)$ by

$$K_h=\big\{v_h|\ v_h\in V_h,\quad v_h(a)\le\varphi(a),\quad\forall a\in\mathcal{N}(\Omega)\cup\mathcal{N}(\partial\Omega)\big\},\tag{3.3}$$

$$\tilde{a}_h(y_h,w_h)=\sum_{\tau\in\mathcal{T}^h}\int_\tau(\alpha\partial_{ij}y_h\cdot\partial_{ij}w_h+y_h\cdot w_h)dx,\quad\forall y_h,w_h\in V_h.\tag{3.4}$$

Under the above definitions, it is obvious that $\|v_h\|_h = \big(\tilde{a}_h(v_h, v_h)\big)^{\frac{1}{2}}$ is a norm on V_h. The Morley nonconforming finite element approximation of problem (3.1) reads

$$\begin{cases} \text{Find } y_h \in K_h, \text{ such that} \\ \tilde{a}_h(y_h, w_h - y_h) \geq (y_0, w_h - y_h), \quad \forall w_h \in K_h. \end{cases} \tag{3.5}$$

Since K_h is clearly a closed convex set of V_h, $\tilde{a}_h(\cdot,\cdot)$ is continuous over $V_h \times V_h$ and V_h-elliptic, problem (3.5) admits a unique solution.

It is well known that y_h is a piecewise quadratic polynomial over triangulation $\mathcal{T}^h$. Recall (1.1), which presents the relation between the state y and the control u, and we can set that $u_h|_\tau = -\Delta y_h|_\tau$, $\forall \tau \in \mathcal{T}^h$. Then the state-constrained optimal control problem can be solved by approximating the variational inequality using the nonconforming finite element scheme (3.5).

4. Error Analysis

In order to analyze the error of the finite element approximation, we first presented known regularities of the solution of the problem (1.1) under some conditions. The regularities of the optimal state y and control u have been discussed by Bergounioux and Kunisch in [5] under the following assumptions:

Assumption (A):

$$\begin{aligned} &\Omega^0 = \bigcup_{i=1}^{l} A_i, \ \bar{\mathring{A}}_i = A_i, \ \Omega^0 \cap \partial\Omega = \emptyset, \\ &A_i, \ i = 1, \cdots, l \text{ are pairwise disjoint}, \\ &A_i \text{ is connected with } \mathscr{C}^{1,1} \text{ boundary for each } i, \end{aligned}$$

Assumption (B):

$$\begin{aligned} &\Omega \subset \mathbb{R}^2, \text{ and } \Omega^0 \text{ is a Lipschitzian, strongly} \\ &\text{non-self-intersecting curve in } \Omega \text{ with } \Omega^0 \cap \partial\Omega = \emptyset, \end{aligned}$$

where the coincidence set Ω^0 is defined in (2.4).

Let the first order optimality system of state-constrained optimal control problem (1.1) be defined by (1.3), then we have (see [5]) the following lemmas:

Lemma 4.1. *Assume that* (*A*) *holds. Then* $p \in H_0^1(\Omega)$, $p|_{\Omega^0} \in H^2(\Omega^0)$, $p|_{\Omega^+} \in H^2(\Omega^+)$.

Lemma 4.2. *Assume that* (*B*) *holds. Then* $p \in W_0^{1,s}(\Omega)$ *for every* $s \in (1, 2)$.

From the above lemmas we can conclude that: when Assumption (A) holds, we have that $u \in H_0^1(\Omega)$ and $y \in H_0^1(\Omega) \cap H^3(\Omega)$; while when Assumption (B) holds, we have $u \in W_0^{1,s}(\Omega)$ and $y \in H_0^1(\Omega) \cap W^{3,s}(\Omega)$ for all $s \in (1, 2)$. Using these conclusions on the regularity of the solution, we can derive the following a priori error estimate.

Theorem 4.3. *Let* y *and* y_h *be the solutions of problems* (2.2) *and* (3.5), *respectively. Assume that* $y_0 \in L^2(\Omega)$, $\varphi \in H^4(\Omega)$, *and Assumption* (*A*) *holds. Then we have*

$$\|y - y_h\|_h^2 + \alpha\|u - u_h\|_{0,\Omega}^2 \leq C(h^2 + \alpha^{-1}h^4), \tag{4.1}$$

where $u = -\Delta y$ *is the solution of* (1.1), *and* $u_h = -\Delta y_h$ *on all the element* $\tau \in \mathcal{T}^h$.

Proof. Recalling the definition of Π_h defined in Section 3, we have that $\Pi_h y \in K_h$. Then it follows from the definition of $\|\cdot\|_h$ and the inequality (3.5) that

$$\begin{aligned}
&\|\Pi_h y - y_h\|_h^2 \\
=& \tilde{a}_h(\Pi_h y - y_h, \Pi_h y - y_h) \\
=& \tilde{a}_h(\Pi_h y - y, \Pi_h y - y_h) + \tilde{a}_h(y - y_h, \Pi_h y - y_h) \\
\leq& \|\Pi_h y - y\|_h \, \|\Pi_h y - y_h\|_h + \tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\Pi_h y - y_h\|_h \\
\leq& \|\Pi_h y - y\|_h + \frac{\tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h)}{\|\Pi_h y - y_h\|_h}.
\end{aligned}$$

Using the triangle inequality we can derive that

$$\begin{aligned}
&\|y - y_h\|_h \\
\leq& C\|y - \Pi_h y\|_h + C\frac{\tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h)}{\|\Pi_h y - y_h\|_h}.
\end{aligned} \tag{4.2}$$

By the standard interpolation error estimate [10, 23] we have

$$\|y - \Pi_h y\|_h \leq Ch|y|_3. \tag{4.3}$$

In the following, we will analyze the term $\tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h)$. Letting $w_h = \Pi_h y - y_h$, we have

$$\begin{aligned}
&\tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h) \\
=& \sum_\tau \int_\tau (\alpha \partial_{ij} y \cdot \partial_{ij} w_h + y \cdot w_h) dx - \int_\Omega y_0 \cdot w_h dx \\
=& \sum_\tau \int_\tau \alpha \Delta y \cdot \Delta w_h dx + \int_\Omega (y - y_0) \cdot w_h dx \\
&\quad + \sum_\tau \int_{\partial\tau} \alpha \partial_{ns} y \cdot \partial_s w_h ds - \sum_\tau \int_{\partial\tau} \alpha \partial_{ss} y \cdot \partial_n w_h ds \\
=:& I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\partial_n y &= \sum_i \partial_i y \cdot n_i, & \partial_s y &= \sum_i \partial_i y \cdot s_i, \\
\partial_{ns} y &= \sum_{ij} \partial_{ij} y \cdot n_i s_j, & \partial_{ss} y &= \sum_{ij} \partial_{ij} y \cdot s_i s_j,
\end{aligned}$$

and $n = (n_1, n_2)$, $s = (s_1, s_2)$ are the unit vectors of the outward normal and the anticlockwise tangential directions, respectively.

By the standard nonconforming finite element error analysis techniques [10, 23] we have

$$\begin{aligned}
I_3 =&\sum_\tau \int_{\partial\tau} \alpha\partial_{ns} y \cdot \partial_s w_h ds \\
=& \sum_{l\cap\partial\Omega=\emptyset} \int_l \alpha\partial_{ns} y \cdot [\partial_s w_h] ds + \sum_{l\cap\partial\Omega\neq\emptyset} \int_l \alpha\partial_{ns} y \cdot \partial_s w_h ds \\
=& \alpha \sum_{l\cap\partial\Omega=\emptyset} \int_l (\partial_{ns} y - \overline{\partial_{ns} y}) \cdot ([\partial_s w_h] - \overline{[\partial_s w_h]}) ds \\
&+ \alpha \sum_{l\cap\partial\Omega\neq\emptyset} \int_l (\partial_{ns} y - \overline{\partial_{ns} y}) \cdot (\partial_s w_h - \overline{\partial_s w_h}) ds \\
\leq& C\alpha \sum_l \sum_{l\subset\partial\tau} h^{\frac{1}{2}} \|\partial_{ns} y\|_{1,\tau} h^{\frac{1}{2}} \|\partial_s w_h\|_{1,\tau} \leq Ch|y|_3 \|w_h\|_h,
\end{aligned} \tag{4.5}$$

where $[v]_l$ is the jump of v on the edge l, while $\bar{v}$ denotes the integral average of v on the edge l. In (4.5), we have used the fact that $\int_l [\partial_s w_h] = 0$ for all the edges because that $w_h \in V_h$ and hence $[w_h] = 0$ on all the nodes. Similarly, it can be deduced that

$$\begin{aligned}
I_4 =& -\sum_\tau \int_{\partial\tau} \alpha\partial_{ss} y \cdot \partial_n w_h \, ds \\
=& -\sum_{l\cap\partial\Omega=\emptyset} \int_l \alpha\partial_{ss} y \cdot [\partial_n w_h] \, ds - \sum_{l\cap\partial\Omega\neq\emptyset} \int_l \alpha\partial_{ss} y \cdot \partial_n w_h \, ds,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{l\cap\partial\Omega=\emptyset} \int_l \alpha\partial_{ss} y \cdot [\partial_n w_h] \, ds \\
=& \alpha \sum_{l\cap\partial\Omega=\emptyset} \int_l \left(\partial_{ss} y - \overline{\partial_{ss} y}\right) \cdot \left([\partial_n w_h] - \overline{[\partial_n w_h]}\right) ds \\
\leq& Ch|y|_3 \|w_h\|_h.
\end{aligned}$$

Note that $\partial_{ss} y|_{\partial\Omega} = 0$ because $y \in H_0^1(\Omega)$. Therefore we have that

$$|I_4| \leq Ch|y|_3 \|w_h\|_h. \tag{4.6}$$

As to I_1, we have

$$\begin{aligned}
I_1 =&\sum_\tau \int_\tau \alpha\Delta y \cdot \Delta w_h \, dx \\
=& -\sum_\tau \int_\tau \alpha\nabla(\Delta y) \cdot \nabla w_h + \sum_\tau \int_{\partial\tau} \alpha\Delta y \cdot \frac{\partial w_h}{\partial n} \, ds.
\end{aligned} \tag{4.7}$$

Note that $\Delta y|_{\partial\Omega} = 0$. Then similar to I_4, it can be derived that

$$\left| \sum_\tau \int_{\partial\tau} \alpha\Delta y \cdot \frac{\partial w_h}{\partial n} \, ds \right| \leq Ch|y|_3 \|w_h\|_h. \tag{4.8}$$

Let w^I be the conventional piecewise linear Lagrange interpolation of w, then we have

$$\begin{aligned}
&-\sum_\tau\int_\tau \alpha\nabla(\Delta y)\cdot\nabla w_h dx + I_2\\
=&-\sum_\tau\int_\tau \alpha\nabla(\Delta y)\cdot\nabla(w_h - w_h^I) + \int_\Omega (y-y_0)\cdot(w_h - w_h^I)dx\\
&-\int_\Omega \alpha\nabla(\Delta y)\cdot\nabla w_h^I dx + \int_\Omega (y-y_0)\cdot w_h^I dx.
\end{aligned}\tag{4.9}$$

By the standard interpolation error estimates we have

$$\Big|\sum_\tau\int_\tau \alpha\nabla(\Delta y)\cdot\nabla(w_h - w_h^I)dx\Big| \le Ch|y|_3\|w_h\|_h,\tag{4.10}$$

and

$$\begin{aligned}
&\Big|\int_\Omega (y-y_0)\cdot(w_h - w_h^I)dx\Big|\\
&\le Ch^2\|y-y_0\|_{0,\Omega}\Big(\sum_\tau \|w_h|_{2,\tau}^2\Big)^{\frac12}\\
&\le C\alpha^{-\frac12}h^2\big(\|y\|_{0,\Omega} + \|y_0\|_{0,\Omega}\big)\|w_h\|_h.
\end{aligned}\tag{4.11}$$

Note that $w^I\in H_0^1(\Omega)\cap C(\bar\Omega)$ and denote $R(y) := \alpha\Delta^2 y + y - y_0$. We have

$$\begin{aligned}
&-\int_\Omega \alpha\nabla(\Delta y)\cdot\nabla w_h^I dx + \int_\Omega (y-y_0)\cdot w_h^I dx\\
=&\int_\Omega R(y)\cdot w_h^I dx = \int_\Omega R(y)\cdot(\Pi_h y - y_h)^I dx\\
=&\int_\Omega R(y)\cdot(\Pi_h y - y)^I dx + \int_\Omega R(y)\cdot(y-y_h)^I dx\\
=&\int_\Omega R(y)\cdot(y-y_h)^I dx\\
=&\int_\Omega R(y)\cdot(y-\varphi)^I dx + \int_\Omega R(y)\cdot(\varphi - y_h)^I dx,
\end{aligned}\tag{4.12}$$

where we have used the fact that $(\Pi_h y - y)^I = 0$ in (4.12). Since $y_h\in K_h$, we have $(\varphi - y_h)^I\ge 0$. Moreover note that $R(y) = \alpha\Delta^2 y + y - y_0 \le 0$. Therefore,

$$\int_\Omega R(y)\cdot(\varphi - y_h)^I dx \le 0.\tag{4.13}$$

It follows from Lemma 2.1 that

$$\alpha\Delta^2 y + y - y_0 = 0 \text{ in } \Omega^+,\quad \alpha\Delta^2 y + y = \alpha\Delta^2\varphi + \varphi \text{ in } \Omega^0.$$

Using the fact $(\alpha\Delta^2 y + y - y_0)\cdot(y-\varphi) = 0$ we have that

$$\begin{aligned}
&\int_\Omega R(y)\cdot(y-\varphi)^I dx\\
=&\int_{\Omega^0} R(y)\cdot(y-\varphi)^I dx = \int_{\Omega^0} R(y)\cdot\big((y-\varphi)^I - (y-\varphi)\big)dx\\
\le& Ch^2\Big(\|y_0\|_{0,\Omega} + \|\Delta^2\varphi\|_{0,\Omega} + \|\varphi\|_{0,\Omega}\Big)\Big(|y|_{2,\Omega} + |\varphi|_{2,\Omega}\Big).
\end{aligned}\tag{4.14}$$

Combining (4.7)-(4.14) we have

$$\begin{aligned} I_1 + I_2 \le\ & Ch|y|_3\|w_h\|_h + C\alpha^{-\frac{1}{2}}h^2\Big(\|y\|_{0,\Omega} + \|y_0\|_{0,\Omega}\Big)\|w_h\|_h \\ & + Ch^2\Big(\|y_0\|_{0,\Omega} + \|\Delta^2\varphi\|_{0,\Omega} + \|\varphi\|_{0,\Omega}\Big)\Big(|y|_{2,\Omega} + |\varphi|_{2,\Omega}\Big). \end{aligned} \tag{4.15}$$

Then (4.4)-(4.6) and (4.15) imply that

$$\begin{aligned} & \tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h) \\ \le\ & Ch|y|_3\|w_h\|_h + C\alpha^{-\frac{1}{2}}h^2\Big(\|y\|_{0,\Omega} + \|y_0\|_{0,\Omega}\Big)\|w_h\|_h \\ & + Ch^2\Big(\|y_0\|_{0,\Omega} + \|\Delta^2\varphi\|_{0,\Omega} + \|\varphi\|_{0,\Omega}\Big)\Big(|y|_{2,\Omega} + |\varphi|_{2,\Omega}\Big). \end{aligned}$$

If $h(|y|_{2,\Omega} + |\varphi|_{2,\Omega}) \le \|w_h\|_h$, then

$$\begin{aligned} & \tilde{a}_h(y, \Pi_h y - y_h) - (y_0, \Pi_h y - y_h) \\ \le\ & Ch\Big(\|y\|_{3,\Omega} + \|\Delta^2\varphi\|_{0,\Omega} + \|\varphi\|_{0,\Omega} + \|y_0\|_{0,\Omega}\Big)\|w_h\|_h \\ & +\ C\alpha^{-\frac{1}{2}}h^2\Big(\|y\|_{0,\Omega} + \|y_0\|_{0,\Omega}\Big)\|w_h\|_h. \end{aligned} \tag{4.16}$$

Then it follows from (4.2), (4.3) and (4.16) that

$$\|y - y_h\|_h \le Ch\Big(\|y\|_{3,\Omega} + \|\varphi\|_{4,\Omega} + \|y_0\|_{0,\Omega}\Big) + C\alpha^{-\frac{1}{2}}h^2\Big(\|y\|_{0,\Omega} + \|y_0\|_{0,\Omega}\Big). \tag{4.17}$$

Otherwise we have

$$\begin{aligned} \|y - y_h\|_h \le & \|y - \Pi_h y\|_h + \|w_h\|_h \\ & \le Ch|y|_3 + Ch\big(|y|_{2,\Omega} + |\varphi|_{2,\Omega}\big) \\ & \le Ch\big(\|y\|_{3,\Omega} + |\varphi|_{2,\Omega}\big). \end{aligned} \tag{4.18}$$

Summing up, we conclude from (4.17) and (4.18) that

$$\|y - y_h\|_h \le Ch + C\alpha^{-\frac{1}{2}}h^2. \tag{4.19}$$

From the definition of $\|\cdot\|_h$ and the fact that $u = -\Delta y$ and $u_h = -\Delta y_h$ on all the elements, we have

$$\alpha\|u - u_h\|_{0,\Omega}^2 \le \|y - y_h\|_h^2 \le Ch^2 + C\alpha^{-1}h^4. \tag{4.20}$$

Thus (4.1) follows from (4.19) and (4.20). □

Remark 4.4. It follows from Theorem 4.3 that

$$\|y - y_h\|_h^2 + \alpha\|u - u_h\|_{0,\Omega}^2 \le Ch^2,$$

if $\alpha \ge Ch^2$. Moreover, if $\alpha \ge Ch^\gamma$, $0 \le \gamma \le 2$, we have

$$\|u - u_h\|_{0,\Omega} \le Ch^{1-\frac{\gamma}{2}}.$$

Remark 4.5. In Theorem 4.3 and Remark 4.4, we provided the error estimate of the finite element approximation under the assumption of the regularity of the solution, i.e., $y \in H^3(\Omega)$. Similar to [13], we can prove the convergence result under weaker regularity conditions. Let y and y_h be the solutions of problems (2.2) and (3.5), respectively. Assume that $y_0 \in L^2(\Omega)$, the state $y \in V$, the control $u \in L^2(\Omega)$, $\varphi \in H^2(\Omega)$, and $\varphi > 0$ on $\partial\Omega$. It can be proven that

$$\lim_{h\to 0} \|y - y_h\|_h = 0, \quad \lim_{h\to 0} \|u - u_h\|_{0,\Omega} = 0.$$

Furthermore, when y has further regularity, i.e., $y \in W^{3,p}(\Omega)$, $1 < p < 2$, as Lemma 4.2, it can be proven that

$$\|y - y_h\|_h^2 + \alpha\|u - u_h\|_{0,\Omega}^2 \leq Ch^{4(1-\frac{1}{p})} + C\alpha^{-1}h^4,$$

where we used the results of the embedding theorem: $W^{3,p}(\Omega) \hookrightarrow H^r(\Omega)$, $r = 2(1 - 1/p)$, and $H^s(\Omega) \hookrightarrow W^{1,q}(\Omega)$, $q = p/(p-1)$, $s = 2 - 2/q = 2/p$.

5. Numerical Examples

In this section, we will present some numerical examples to illustrate our new approach.

5.1. Numerical algorithms

At first, we will give two numerical algorithms for the fourth order variational inequality. The first one is the projected SOR algorithm proposed in [1] (the similar over-relaxation method can be founded in [12]), which can be stated as:

Algorithm 5.1. *Let $\mathcal{K}$ denote the unions of internal nodes, $0 < \omega < 2$, δ be a given tolerance, and initial value $y_h^0 \in V_h$.*
(a) Set $y_h^0 := min\{y_h^0, \varphi\}$, and $k = 1$.
(b) Set $\tilde{\mathcal{K}} := \mathcal{K}$ and $y_h^k = y_h^{k-1}$.
(i) Choose $z \in \tilde{\mathcal{K}}$.
(ii) Compute the minimize t^* for $J_h(y_h^k + t\psi_z)$ among all $t \in \mathbb{R}$ such that $y_h^k + t\psi_z \in K_h$, where ψ_z is the basis function at nodal z.
(iii) Set $y_h^{k+1} := P_{K_h}(y_h^k + \omega t^* \psi_z)$, where $P_{K_h}(y)$ denotes the projection of y on K_h.
(iv) Set $\tilde{\mathcal{K}} =: \tilde{\mathcal{K}} \setminus \{z\}$ and go to (i) if $\tilde{\mathcal{K}} \neq \emptyset$.
(c) Set $y_h = y_h^{k+1}$ and stop if $\|y_h^{k+1} - y_h^k\|_{0,2,\Omega} \leq \delta$.
(d) Set $k =: k + 1$ and go to (b).

Next, we will introduce another widely used algorithm known as the dual iterative method for solving fourth order variational inequalities (2.2), which has been discussed in [12] in details. We first consider the dual iterative method for continuous problem (1.3). When $y \in H^2(\Omega) \cap H_0^1(\Omega)$, $\varphi \in L^2(\Omega)$, it is reasonable to define

$$\Lambda = \big\{\mu \,|\mu \in L^2(\Omega), (\mu, v)_{L^2(\Omega)} \geq 0 \ \forall v \in L^2(\Omega), \ v \geq 0 \ a.e. \text{ in } \Omega\big\}. \tag{5.1}$$

Following Glowinski ([12]), we introduce the Lagrangian functional $\mathscr{L} : V \times L^2(\Omega) \to \mathbb{R}$ associated with problem (1.3) defined by

$$\mathscr{L}(y, \mu) = J(y) + \int_\Omega \mu \cdot (y - \varphi) dx, \tag{5.2}$$

where $J(y)$ is the minimization functional

$$\frac{1}{2}\int_{\Omega}(y-y_0)^2dx+\frac{\alpha}{2}\int_{\Omega}|\Delta y|^2dx. \tag{5.3}$$

Now the duality algorithm can be described as follows.

Algorithm 5.2': *Given a tolerance TOL and a parameter $\rho\in(0,1)$*
(i) $\lambda^0\in\Lambda$ is chosen arbitrarily;
(ii) For $\lambda^n\in\Lambda$, calculate y^n and λ^{n+1} by means of

$$\begin{aligned}
&y^n\in V=H^2(\Omega)\cap H_0^1(\Omega); && (5.4)\\
&a(y^n,v)=(y_0,v)-(\lambda^n,v),\quad \forall\ v\in V, && (5.5)\\
&\lambda^{n+1}=P_\Lambda\big(\lambda^n+\rho(y^n-\varphi)\big),\quad \rho>0, && (5.6)
\end{aligned}$$

where $a(\cdot,\cdot)$ is defined in Section 2 and P_Λ is a projection operator on Λ;
(iii) Calculate $Error(y)=\|y^n-y^{n-1}\|_{L^2(\Omega)}$ and $Error(\lambda)=\|\lambda^{n+1}-\lambda^n\|_{L^2(\Omega)}$, output $y=y^n$ and stop if $Error(y)+Error(\lambda)\le TOl$;
(iv) Set $n=:\ n+1$ and go to (ii).

In the next step we consider the discrete variants of Algorithm 5.2, which takes the nonconforming finite element approximation into account.

Analogous to (5.1)-(5.2), we define the Lagrangian functional $\mathscr{L}_h: V_h\times\Lambda_h\to\mathbb{R}$ by

$$\mathscr{L}_h(v_h,\mu_h)=J_h(v_h)+\sum_{\tau\in\mathcal{T}^h}\int_\tau \mu_h(v_h-\varphi)dx, \tag{5.7}$$

where J_h is defined as follows:

$$J_h(y_h)=\frac{1}{2}\int_{\Omega}(y_h-y_0)^2dx+\frac{\alpha}{2}\sum_{\tau\in\mathcal{T}^h}\int_\tau\big|\partial_{ij}y_h\partial_{ij}y_h\big|dx,$$

and

$$\Lambda_h=\big\{\mu_h\ |\mu_h\in L^2(\Omega),\ \mu_h|_\tau\in P_1,\ \forall\tau\in\mathcal{T}^h,\ \mu_h(a)\ge 0,\quad \forall a\in\mathcal{N}(\Omega)\cup\mathcal{N}(\partial\Omega)\big\}. \tag{5.8}$$

Then the discrete variant of Algorithm 5.2' reads:

Algorithm 5.2. *Given a tolerance TOL and a parameter $\rho\in(0,1)$*
(i) $\lambda_h^0\in\Lambda_h$ is chosen arbitrarily;
(ii) For $\lambda_h^n\in\Lambda_h$, calculate y_h^n and λ_h^{n+1} by means of

$$\begin{aligned}
&y_h^n\in V_h,\ \tilde a_h(y_h^n,v_h)=(y_0,v_h)-(\lambda_h^n,v_h),\ \forall\ v_h\in V_h, && (5.9)\\
&\lambda_h^{n+1}=P_{\Lambda_h}(\lambda_h^n+\rho(y_h^n-\varphi)),\ \rho>0, && (5.10)
\end{aligned}$$

where $\tilde a_h(\cdot,\cdot)$ is defined in (3.4) and P_{Λ_h} is a projection operator on Λ_h;
(iii) Calculate $Error(y_h)=\|y_h^n-y_h^{n-1}\|_{L^2(\Omega)}$ and $Error(\lambda_h)=\|\lambda_h^{n+1}-\lambda_h^n\|_{L^2(\Omega)}$, output $y_h=y_h^n$ and stop if $Error(y_h)+Error(\lambda_h)\le TOl$;
(iv) Set $n=:\ n+1$ and go to (ii).

Remark 5.1. We have stated two algorithms in this section. Algorithm 5.1 (SOR algorithm) is simpler and easier to implement. Algorithm 5.2 (the dual iterative method) is a standard method for solving fourth order variational inequalities. For example, when the constrained set K is complicated, e.g., it involves the gradient of state y, the projection operator $P_{K_h}(y)$ in Algorithm 5.1 will be difficult to construct, while Algorithm 5.2 is more flexible to deal with various constrained sets.

5.2. Numerical examples

In this section, we will illustrate some computational results solved by using Algorithm 5.1 provided in the last subsection. Numerical results obtained using Algorithm 5.2 were found to be similar, and so have been omitted here.

Firstly, we consider the following three examples that come from Examples 5.1-5.3 in [4] and [15]. Let us consider the following state-constrained optimal control problem:

$$\min_{y\in K}\left\{\frac{1}{2}\int_\Omega (y-y_0)^2dx+\frac{\alpha}{2}\int_\Omega u^2dx\right\} \tag{5.11}$$

subject to

$$-\Delta y = u \quad \text{in } \Omega, \qquad y=0 \text{ on } \partial\Omega,$$

with

$$K=\big\{v\in H_0^1(\Omega):\ v\le\varphi \ a.e. \text{ in } \Omega=[0,1]\times[0,1]\big\}.$$

Then the three numerical examples are:

Example 5.1. The optimal control problem (5.11) with the data $\alpha = 0.1$, $\varphi = 0.01$ and $y_0(x,y) = 10\big(\sin(2x)+y\big)$.

Example 5.2. The optimal control problem (5.11) with the data $\alpha = 10^{-3}$, $\varphi = 0.1$ and $y_0(x,y) = \sin(2\pi xy)$.

Example 5.3. The optimal control problem (5.11) with the data $\alpha = 10^{-4}$, $\varphi = 1$ and $y_0(x,y) = \sin(4\pi xy) + 1.5$.

In computing Examples 5.1-5.3, we adopt the relaxation parameter $\omega = 1.5$ and the tolerance $\delta = 10^{-3}$. The number of the elements is 4096. Note that there are no known exact solutions for the three examples, so it is impossible for us to show the computational error. We thus just present the figures of the numerical solutions for Examples 5.1-5.3 in Figures 5.1-5.3. Because we use the Morley nonconforming finite element to approximation the state y, the approximation state y_h is a discontinuous piecewise quadratic polynomial. We show y_h in the figures using the piecewise linear interpolation of y_h just for simplicity. The discrete control u_h was computed by the relation $u_h|_\tau = -\Delta y_h|_\tau$, $\forall\tau\in\mathcal{T}^h$. It is clear that u_h is piecewise constant.

By comparing the figures of y_h and u_h above with those shown in [15], it can be concluded that the numerical results using our new approach are similar to those obtained by their methods, and that our numerical results seem to be reasonable.

Example 5.4. In this example we consider the following example with the objective functional:

$$\min_{y\in K}\left\{\frac{1}{2}\int_\Omega (y-y_0)^2dx+\frac{\alpha}{2}\int_\Omega (u-u_0)^2dx\right\}.$$

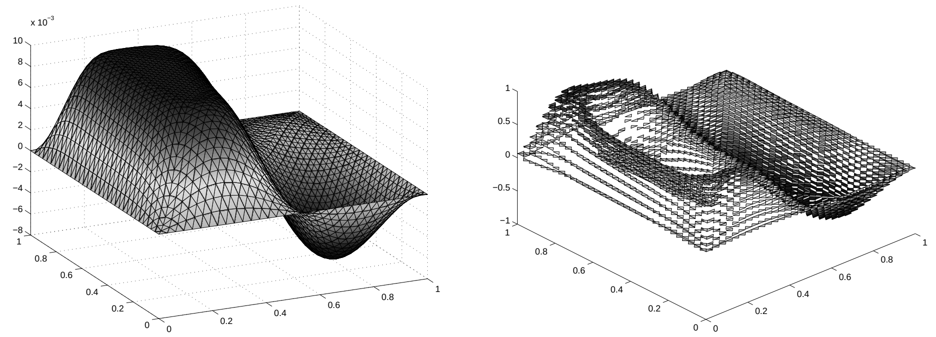

Fig. 5.1. The numerical state y_h(left) and control u_h(right) of Example 5.1.

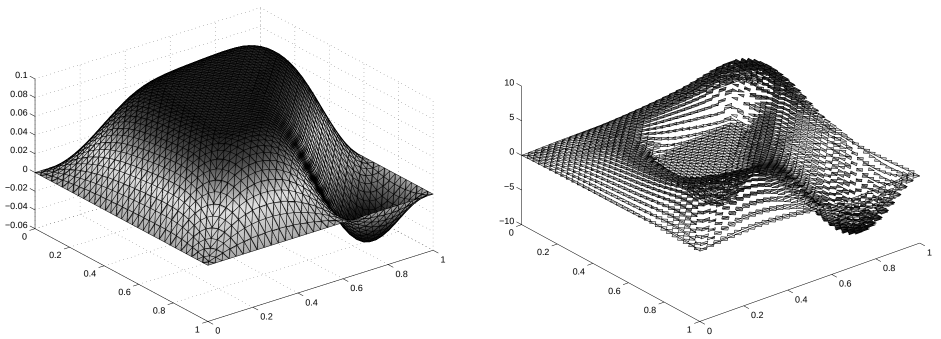

Fig. 5.2. The numerical state y_h(left) and control u_h(right) of Example 5.2.

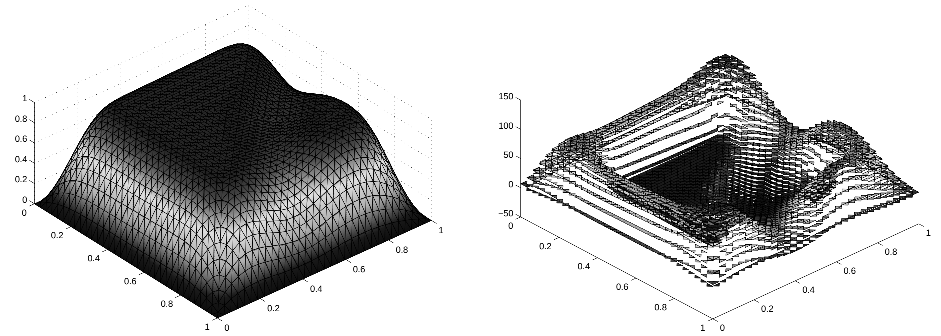

Fig. 5.3. The numerical state y_h(left) and control u_h(right) of Example 5.3.

This example is a slight modification of an example in [17] so that we know the exact solution. The data of the problem are as follows and $\Omega =: [-2,2]\times[-2,2]$,

$$y_0 = y(r) + \Delta p(r) + \lambda(r),\ u_0 = u(r) + \alpha^{-1}p(r),\ \psi = 0,\ \alpha = 0.1,$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\forall (x_1, x_2) \in \Omega$, and $y(r),\ u(r),\ p(r)$ are chosen according to

$$\begin{aligned} y(r) &= -r^2\gamma_1(r), \quad u(r) = -\Delta y(r), \\ p(r) &= \gamma_2(r)\left(r^4 - \frac{3}{2}r^3 + \frac{16}{9}r^2\right), \end{aligned}$$

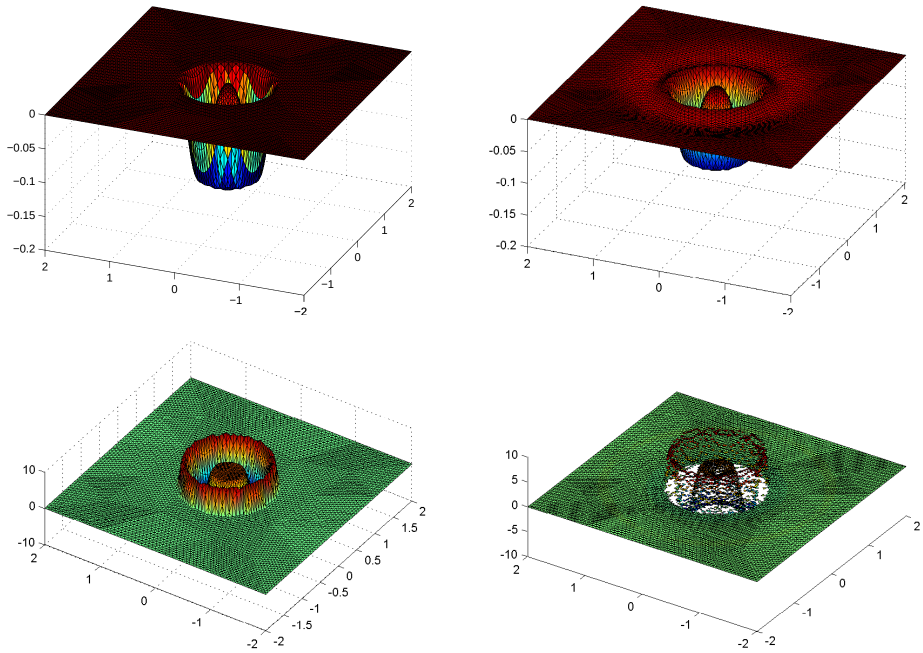

Fig. 5.4. Example 5.4: the exact state y (top left) and numerical state y_h (top right); the exact control u (bottom left) and numerical control u_h (bottom right).

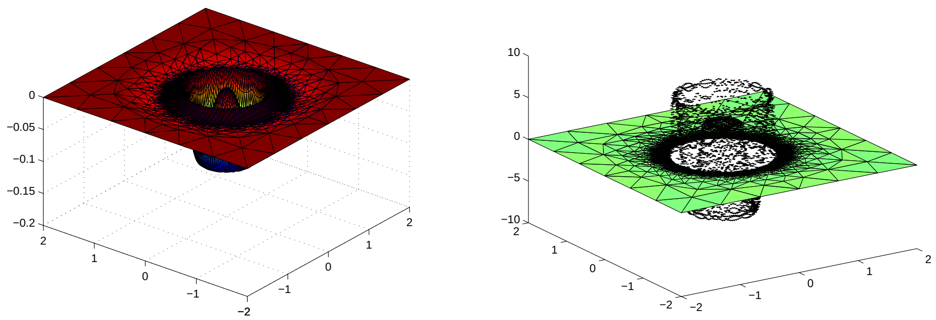

Fig. 5.5. The numerical state y_h(left) and control u_h of Example 5.4 on local refined meshes.

where

$$\gamma_1(r) = \begin{cases} 1, & r < 0.25, \\ -192(r-0.25)^5 + 240(r-0.25)^4 - 80(r-0.25)^3 + 1, & 0.25 < r < 0.75, \\ 0, & \text{otherwise}, \end{cases}$$

$$\gamma_2(r) = \begin{cases} 1, & r < 0.75, \\ 0, & \text{otherwise}, \end{cases} \qquad \lambda(r) = \begin{cases} 0, & r < 0.75, \\ 0.1, & \text{otherwise}. \end{cases}$$

The figures of the exact solutions and the numerical solutions for the state y and the control u are shown in Figures 5.4, respectively. In Table 5.1, we list the errors of the state y and the control u, where the norm $\|\cdot\|_h$ is defined in Section 3, and N denotes the number of nodes of the triangulation. Noting that in this example, $y\bar{\in}H^3(\Omega)$ $\big($we only have $y \in H^2(\Omega)\big)$, the requirements of Theorem 4.3 are not satisfied. But it can be seen from Table 5.1 that when the

Table 5.1: Error of the state y and control u on uniform meshes.

N	113	417	945	1601	3681	6273
$\|y-y_h\|_h$	1.5380	1.1449	0.9637	0.7208	0.5928	0.4248
$\|u-u_h\|_0$	1.5362	1.1447	0.9634	0.7205	0.5924	0.4235

Table 5.2: Error of the state y and control u on local refined meshes.

N	225	305	553	957	1837	3245
$\|y-y_h\|_h$	1.1446	0.9631	0.7204	0.5931	0.4490	0.3655
$\|u-u_h\|_0$	1.1444	0.9628	0.7201	0.5927	0.4468	0.3640

meshes are fine enough, convergence rate is roughly $\mathcal{O}(N^{-\frac{1}{2}}) = \mathcal{O}(h)$, which coincides with our theoretical analysis.

Since the solution of Example 5.4 has singularities near the free boundary $x_1^2 + x_2^2 = 0.75$, we should make local refinements near the singularities to obtain the better approximations. In the following we show the numerical results both for the state and control on the local refined meshes. It is clear that the computing efficiency can be improved by using the local refinement strategy.

6. Discussion

In this paper, we study new numerical methods for an optimal control problem with pointwise state constraints. We reformulate the state-constrained optimal control into a constrained minimization problems only involving the state, whose optimality condition is characterized by a fourth order elliptic variational inequality. Then direct numerical algorithms are proposed for the inequality, and error estimates of the finite element approximation are derived. Numerical experiments illustrate effectiveness of the new approach. There are many important issues that remain to be studied. For example a posteriori error estimate and adaptive finite element method can be studied by using this approach. They should be able to improve the computing efficiency as shown in our last numerical example.

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