

FINITE VOLUME SCHEME FOR MULTIPLE FRAGMENTATION EQUATIONS

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Abstract. In this paper we study a finite volume approximation for the conservative formulation of multiple fragmentation models. We investigate the convergence of the numerical solutions towards a weak solution of the continuous problem by considering locally bounded kernels. The proof is based on the Dunford-Pettis theorem by using the weak L^1 compactness method. The analysis of the method allows us to prove the convergence of the discretized approximated solution towards a weak solution to the continuous problem in a weighted L^1 space.

Key words. Finite volume, Fragmentation, Convergence, Particle.

1. Introduction

The equations we consider in this paper describe the time evolution of the particle size distribution (PSD) under multiple fragmentation or breakage process. In the simplest equations, each particle is identified by its size, i.e. volume or mass. In multiple breakage, a big particle breaks into two or many fragments. Examples of applications of such models arise in many engineering applications, including aerosol physics, the coalescence and breakup of liquid drops, high shear granulation, crystallization, atmospheric science, highly demanding nano-particles and pharmaceutical industries, see Sommer et al. [16], Gokhale et al. [6] and references therein. Binary breakage is not adequate for some of these applications, therefore, multiple fragmentation is preferred.

The temporal change of the particle number density, $f(t, x) \geq 0$, of particles of volume $x \in \mathbb{R}_{>0}$ at time $t \in \mathbb{R}_{>0}$ in a spatially homogeneous physical system undergoing a breakage process is described by the following well known population balance equation (PBE), see [18]

$$(1) \quad \frac{\partial f(t, x)}{\partial t} = \int_x^\infty b(x, y) S(y) f(t, y) dy - S(x) f(t, x),$$

with initial data

$$(2) \quad f(0, x) = f^{in}(x) \geq 0, \quad x \in]0, \infty[.$$

The positive term on the right-hand side describes the creation of particles of size x when a particle of size y breaks. The negative term explains the disappearance of particles of size x into smaller pieces. These terms are known to be the birth and the death term, respectively. The selection rate $S(y)$ gives the rate at which particles of size y are selected to break. The breakage function $b(x, y)$ for a given $y > 0$ gives the size distribution of particle sizes $x \in [0, y[$ resulting from the breakage of a particle of size y . For the particular case of $b(x, y) = 2/y$, the multiple breakage

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turns into the binary breakage PBE. The breakage function satisfies the following important properties

$$(3) \quad \int_0^x b(u, x) du = \bar{N}(x), \quad \int_0^x ub(u, x) du = x.$$

The function $\bar{N}(x)$, which may be infinite, denotes the number of fragments obtained from the breakage of a particle of size x . The second integral shows that the total mass created from the breakage of a particle of size x is again x .

Besides the information given by the evolution of the particle number density distribution, some integral properties like moments are also of great interest in particulate systems. The j th moment of the particle size distribution is defined as

$$(4) \quad \mu_j(t) = \int_0^\infty x^j f(t, x) dx.$$

The first two moments are of special interest. The zeroth ($j = 0$) and first ($j = 1$) moments are proportional to the total number and total mass of particles respectively. Furthermore, the second moment is proportional to the light scattered by particles in the Rayleigh limit [9, p. 1325], [15, p. 267] in some applications. One can easily show that the zeroth moment increases by breakage process while the total mass stays constant. For the total mass conservation, the integral equality

$$\int_0^\infty xf(t, x) dx = \int_0^\infty xf^{in}(x) dx, \quad t \geq 0,$$

holds.

Several researchers showed the existence of weak solutions for the aggregation-breakage equations with non-increasing mass for a large class of aggregation and fragmentation kernels, see Laurençot [10, 11] and the references therein. Some authors also explained the relationship between discrete and continuous models. For instance, Ziff and McGrady [17] found this relationship for constant and sum breakage kernels while Laurençot and Mischler [11] gave results for the aggregation-breakage models under more general assumptions on the kernels, i.e. for bilinear growth. In the literature, there are various ways to approximate the continuous aggregation-breakage equations including deterministic method [4, 13] and Monte Carlo method [3, 7].

Recently, Bourgade and Filbet [1] have used a finite volume approximation for the binary aggregation-breakage equation. They gave the convergence result of the numerical solutions towards a weak solution of the continuous equation by considering locally bounded kernels. However, their study is restricted to the case of binary breakage. As mentioned above, the case of multiple breakage is of great importance in several applications, especially in high shear granulation. Therefore the aim of this work is to provide a finite volume approximation of the multiple breakage PBE and to investigate its convergence. Though the central idea of this extension is based on the work of Bourgade and Filbet [1], the finite volume approximation and its convergence presented in this work differ due to appearance of completely new kinetics parameters (b and S) in the case of multiple breakage. Following the idea of Bourgade and Filbet, the proof is based on the Dunford-Pettis theorem by using the weak L^1 compactness method and the La Vallée Poussin theorem. We prove

the convergence of the discretized approximated solution towards a weak solution to the continuous problem in the weighted L^1 space X^+ given by

$$X^+ = \{f \in L^1(\mathbb{R}_{>0}) \cap L^1(\mathbb{R}_{>0}, x \, dx) : f \geq 0, \|f\| < \infty\}$$

where $\|f\| = \int_0^\infty (1+x)|f(x)|dx$, for the non-negative initial condition $f^{in} \in X^+$ and $\mathbb{R}_{>0} =]0, \infty[$. Here the notation $L^1(\mathbb{R}_{>0}, x \, dx)$ stands for the space of the Lebesgue measurable real valued functions on $\mathbb{R}_{>0}$ which are integrable with respect to the measure $x \, dx$.

The outline of the paper is as follows. The conservative formulation of the continuous multiple breakage equation, which is needed for further analysis, is discussed in the next section. Section 3 gives the numerical approximation of this equation. Further in Section 4 we discuss the convergence of the approximated solution using weak compactness. Finally, conclusions are made.

2. Conservative formulation

As mentioned earlier, mass is a conserved quantity in the fragmentation phenomena. Therefore, one can also rewrite the equation in a conservative form of mass density $xf(t, x)$ as

$$(5) \quad \frac{x\partial f(t, x)}{\partial t} = \frac{\partial \mathcal{F}(f)}{\partial x}(x), \quad (t, x) \in \mathbb{R}_{>0}^2 =]0, \infty[^2$$

where the continuous flux is given as

$$\mathcal{F}(f)(x) := \int_x^\infty \int_0^x ub(u, v)S(v)f(t, v)du dv, \quad x \in \mathbb{R}_{>0}.$$

The proof relies on applying the Leibnitz integration rule and by using the mass conserving property (3) as

$$\begin{aligned} \frac{x\partial f(t, x)}{\partial t} &= \int_x^\infty \frac{\partial}{\partial x} \int_0^x ub(u, v)S(v)f(t, v) \, du \, dv - \int_0^x ub(u, x)S(x)f(t, x) \, du \\ &= \int_x^\infty xb(x, v)S(v)f(t, v) \, dv - S(x)f(t, x) \int_0^x ub(u, x) \, du \\ (6) \quad &= x \int_x^\infty b(x, v)S(v)f(t, v) \, dv - S(x)f(t, x)x. \end{aligned}$$

Given $f^{in} \in X^+$, we consider the initial condition

$$f(0, x) = f^{in}(x), \quad x \in \mathbb{R}_{>0}.$$

We now present a numerical scheme to solve the equation (5). For this a finite volume discretization is taken with respect to the volume variable x while an explicit Euler method is used to discretize the time variable t . For the analysis, we have assumed that the multiplicative kernel (product of breakage and selection functions) is locally bounded, i.e. $bS \in L_{loc}^\infty(\mathbb{R}_{>0} \times \mathbb{R}_{>0})$. It should be mentioned that the case of binary aggregation can be added here in the same way as discussed by Bourgade and Filbet [1]. The analysis will follow analogously by adding the conservative aggregation flux taken from [1] to our numerical breakage flux.

3. Numerical approximation

The discretization we propose here is to give a mass conservative truncation for the breakage operator: Given a positive real R , it is defined as

$$\mathcal{F}_c^R(f)(x) := \int_x^R \int_0^x ub(u, v)S(v)f(t, v)dudv.$$

Therefore, a conservative formulation for multiple breakage is given by

$$(7) \quad \begin{cases} x \frac{\partial f}{\partial t} = \frac{\partial \mathcal{F}_c^R(f)}{\partial x}(x), & (t, x) \in \mathbb{R}_{>0} \times]0, R]; \\ f(0, x) = f^{in}(x), & x \in]0, R]. \end{cases}$$

Mass conservation can easily be seen by integrating equation (7) with respect to x from 0 to R .

Now, for the volume discretization of equation (7), let $h \in]0, 1[$, I^h a positive integer such that $(x_{i-1/2})_{i \in \{0, \dots, I^h\}}$ is a mesh of $]0, R]$ with the properties

$$x_{-1/2} = 0, \quad x_{I^h+1/2} = R, \quad x_i = (x_{i-1/2} + x_{i+1/2})/2, \quad \Delta x_i = x_{i+1/2} - x_{i-1/2} \leq h$$

and $\Lambda_i^h =]x_{i-1/2}, x_{i+1/2}]$ for $i \geq 0$. For the time discretization, let us assume that Δt denotes the time step such that $N\Delta t = T$ for a large positive integer N and $[0, T]$ is the time domain where we study the equation. We define the time interval

$$\tau_n = [t_n, t_{n+1}[$$

with $t_n = n\Delta t$, $n \geq 0$.

Now we introduce the finite volume method for the equation. We consider the approximation of $f(t, x)$ for $t \in \tau_n$ and $x \in \Lambda_i^h$ as f_i^n for each integer $i \in \{0, \dots, I^h\}$ and each $n \in \{0, \dots, N-1\}$. For the time being we discretize the selection function $S(x)$ and the breakage function $b(u, x)$ in such a way that $S(x) \approx S^h(x) = S_i$ and $b(u, x) \approx b^h(u, x) = b_{j,i}$ for $x \in \Lambda_i^h$ and $u \in \Lambda_j^h$.

Integrating equation (7) with respect to x and t over a cell in space Λ_i^h and time τ_n respectively gives

$$\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial(xf(t, x))}{\partial t} dx dt = \int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \mathcal{F}_c^R(f)}{\partial x}(x) dx dt.$$

This further implies that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (xf(t_{n+1}, x) - xf(t_n, x)) dx = \int_{t_n}^{t_{n+1}} (\mathcal{F}_c^R(f(t, x_{i+1/2})) - \mathcal{F}_c^R(f(t, x_{i-1/2}))) dt.$$

Finally we obtain the following discretization for the multiple breakage equation

$$(8) \quad x_i f_i^{n+1} = x_i f_i^n + \frac{\Delta t}{\Delta x_i} (\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n)$$

where $\mathcal{F}_{i+1/2}^n$ is the numerical flux which is an approximation of the continuous flux function $\mathcal{F}_c^R(f)(x)$. It is defined as

$$\begin{aligned}
 \mathcal{F}_c^R(f)(x_{i+1/2}) &= \int_{x_{i+1/2}}^R \int_0^{x_{i+1/2}} ub(u, v)S(v)f^n(v)du dv \\
 &= \sum_{j=i+1}^{I^h} \int_{\Lambda_j^h} S(v)f^n(v) \sum_{k=0}^i \int_{\Lambda_k^h} ub(u, v)du dv \\
 (9) \qquad \qquad \qquad &\approx \sum_{j=i+1}^{I^h} \sum_{k=0}^i x_k S_j b_{k,j} f_j^n \Delta x_j \Delta x_k = \mathcal{F}_{i+1/2}^n.
 \end{aligned}$$

The initial condition is taken as

$$f_i^{in} = \frac{1}{\Delta x_i} \int_{\Lambda_i^h} f^{in}(x)dx, \quad i \in \{0, \dots, I^h\}.$$

The breakage fluxes at the boundaries $x_{-1/2}$ and $x_{I^h+1/2}$ are

$$(10) \qquad \qquad \qquad \mathcal{F}_{-1/2}^n = \mathcal{F}_{I^h+1/2}^n = 0.$$

For time we use the explicit Euler discretization while for the volume variable a finite volume approach is considered, see LeVeque [14] and Eymard et al. [5]. Let us denote the characteristic function $\chi_A(x)$ of a set A such that $\chi_A(x) = 1$ if $x \in A$ or 0 elsewhere. Then we define a function f^h on $[0, T] \times]0, R]$ as

$$(11) \qquad \qquad \qquad f^h(t, x) = \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} f_i^n \chi_{\Lambda_i^h}(x) \chi_{\tau_n}(t).$$

This implies that the function f^h depends on the time and volume steps and note that

$$f^h(0, \cdot) = \sum_{i=0}^{I^h} f_i^{in} \chi_{\Lambda_i^h}(\cdot)$$

converges strongly to f^{in} in $L^1]0, R]$ as $h \rightarrow 0$. We also define the breakage and selection functions in discrete form as

$$(12) \qquad \qquad \qquad b^h(u, v) = \sum_{j=0}^{I^h} \sum_{i=0}^{I^h} b_{i,j} \chi_{\Lambda_i^h}(u) \chi_{\Lambda_j^h}(v) \quad \text{where} \quad b_{i,j} = \frac{1}{\Delta x_i \Delta x_j} \int_{\Lambda_j^h} \int_{\Lambda_i^h} b(u, v) du dv$$

and

$$(13) \qquad \qquad \qquad S^h(v) = \sum_{i=0}^{I^h} S_i \chi_{\Lambda_i^h}(v) \quad \text{where} \quad S_i = \frac{1}{\Delta x_i} \int_{\Lambda_i^h} S(v) dv.$$

Such discretization ensures that $\|b^h - b\|_{L^1([0,R] \times]0,R])} \rightarrow 0$ and $\|S^h - S\|_{L^1([0,R])} \rightarrow 0$ as $h \rightarrow 0$.

4. Convergence of solutions

In the following we state our main theorem for the convergence of approximated solutions towards a weak solution of the equation (7).

Theorem 4.1. *Let the breakage function b and the selection function S be such that $bS \in L_{loc}^\infty(\mathbb{R}_{>0} \times \mathbb{R}_{>0})$ and $f^{in} \in X^+$. We also assume that there exists a constant $\theta > 0$ such that the time step Δt satisfies the stability condition*

$$(14) \quad C(T, R)\Delta t \leq \theta < 1,$$

where

$$(15) \quad C(T, R) := \|bS\|_{L^\infty R}.$$

Then up to the extraction of a subsequence,

$$f^h \rightarrow f \text{ in } L^\infty(0, T; L^1]0, R]),$$

where f is the weak solution to (7) on $[0, T]$ with initial data f^{in} . Precisely, the function $f \geq 0$ satisfies

$$(16) \quad \int_0^T \int_0^R x f(t, x) \frac{\partial \varphi}{\partial t}(t, x) dx dt + \int_0^R x f^{in}(x) \varphi(0, x) dx = \int_0^T \int_0^R \mathcal{F}_c^R(t, x) \frac{\partial \varphi}{\partial x}(t, x) dx dt$$

for all continuously differentiable functions φ compactly supported in $[0, T[\times]0, R]$.

It is clear from this theorem that our main aim is to show that the sequence of functions $(f^h)_{h \in \mathbb{N}}$ converges weakly to a function f in $L^1]0, R]$ as h and Δt go to zero. The proof relies on the following Dunford-Pettis theorem [2] which gives a necessary and sufficient condition for compactness with respect to the weak convergence in L^1 .

Theorem 4.2. [2, Theorem 3.2] *Let $|\Omega| < \infty$ and $f^h : \Omega \mapsto \mathbb{R}$ be a sequence in $L^1(\Omega)$. Suppose that the sequence $\{f^h\}$ satisfies*

- $\{f^h\}$ is equibounded in $L^1(\Omega)$, i.e.

$$(17) \quad \sup \|f^h\|_{L^1(\Omega)} < \infty$$

- $\{f^h\}$ is equiintegrable, iff

$$(18) \quad \int_\Omega \Phi(|f^h|) dx < \infty$$

for some increasing function $\Phi : [0, \infty[\mapsto [0, \infty[$ satisfying

$$\lim_{r \rightarrow \infty} \frac{\Phi(r)}{r} \rightarrow \infty.$$

Then f^h lies in a weakly compact set in $L^1(\Omega)$ which implies that there exists a subsequence of f^h that converges weakly in $L^1(\Omega)$.

Therefore, in order to prove the Theorem 4.1, we must show the equiboundedness and the equiintegrability of the family f^h in L^1 as in (17) and (18), respectively. In the following proposition, we prove the non-negativity and equiboundedness of the function f^h . For this we use a mid-point approximation of a point x by $X^h(x)$, i.e. $X^h(x) = x_i$ for $x \in \Lambda_i^h$.

Proposition 4.3. *Let us assume that the time step Δt satisfies (14). Then f^h is a non-negative function satisfying the mass conservation*

$$\int_0^R X^h(x) f^h(t, x) dx = \int_0^R X^h(x) f^h(s, x) dx, \quad 0 \leq s \leq t \leq T$$

and for all $t \in [0, T]$,

$$(19) \quad \int_0^R f^h(t, x) dx \leq \|f^{in}\|_{L^1} e^{R\|bS\|_{L^\infty} t}.$$

Proof. We prove the non-negativity and equiboundedness of f^h by using induction. We know that at $t = 0$, $f^h(0) \geq 0$ and belongs to $L^1]0, R]$. Assume next that the function $f^h(t^n) \geq 0$ and

$$(20) \quad \int_0^R f^h(t^n, x) dx \leq \|f^{in}\|_{L^1} e^{R\|bS\|_{L^\infty} t^n}.$$

Now we will prove that $f^h(t^{n+1}) \geq 0$. We do this first for the cell at the boundary which has the index $i = 0$. Note that by (9) we have $\mathcal{F}_{i \pm 1/2}^n \geq 0$. Therefore, in this case from the equation (8) and by using the flux $\mathcal{F}_{-1/2}^n = 0$, we get

$$x_0 f_0^{n+1} = x_0 f_0^n + \frac{\Delta t}{\Delta x_0} \mathcal{F}_{1/2}^n \geq x_0 f_0^n.$$

Hence we obtain $f_0^{n+1} \geq 0$. Now for $i \geq 1$,

$$x_i f_i^{n+1} = x_i f_i^n + \frac{\Delta t}{\Delta x_i} \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right).$$

From the equation (9) and the non-negativity of $f^h(t^n)$, we calculate

$$\begin{aligned} \frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{\Delta x_i} &= \frac{1}{\Delta x_i} \left[\sum_{j=i+1}^{I^h} \sum_{k=0}^i x_k S_j b_{k,j} f_j^n \Delta x_j \Delta x_k - \sum_{j=i}^{I^h} \sum_{k=0}^{i-1} x_k S_j b_{k,j} f_j^n \Delta x_j \Delta x_k \right] \\ &= \frac{1}{\Delta x_i} \left[- \sum_{k=0}^{i-1} x_k S_i b_{k,i} f_i^n \Delta x_i \Delta x_k + \sum_{j=i+1}^{I^h} x_i S_j b_{i,j} f_j^n \Delta x_j \Delta x_i \right] \\ (21) \quad &\geq - \sum_{k=0}^{i-1} x_k S_i b_{k,i} f_i^n \Delta x_k. \end{aligned}$$

Since $k < i$ implies that $x_k < x_i$, we further simplify (21) into

$$\begin{aligned} \frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{\Delta x_i} &\geq - \sum_{k=0}^{i-1} x_i S_i b_{k,i} f_i^n \Delta x_k \\ &\geq - \sum_{k=0}^{I^h} (S_i b_{k,i} \Delta x_k) x_i f_i^n. \end{aligned}$$

Therefore, we estimate that

$$x_i f_i^{n+1} \geq \left(1 - \Delta t \sum_{k=0}^{I^h} S_i b_{k,i} \Delta x_k \right) x_i f_i^n.$$

Finally, using the stability condition (14) on the time step Δt and the L^1 estimate (20) give

$$f^h(t^{n+1}) \geq 0.$$

Next, the total mass conservation follows by summing (8) with respect to i and using (10)

$$\sum_{i=0}^{I^h} \Delta x_i x_i f_i^{n+1} = \sum_{i=0}^{I^h} \Delta x_i x_i f_i^n + \Delta t \sum_{i=0}^{I^h} \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right) = \sum_{i=0}^{I^h} \Delta x_i x_i f_i^n.$$

Now, we prove that $f^h(t^{n+1})$ enjoys a similar estimate as (20). Multiplying equation (8) by $\Delta x_i/x_i$ and taking summation over i yield

$$\sum_{i=0}^{I^h} \Delta x_i f_i^{n+1} = \sum_{i=0}^{I^h} \Delta x_i f_i^n + \Delta t \sum_{i=0}^{I^h} \frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{x_i}.$$

Analogously as for (21) we may estimate

$$(22) \quad \sum_{i=0}^{I^h} \frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{x_i} \leq \sum_{i=0}^{I^h} \Delta x_i \sum_{j=i+1}^{I^h} \Delta x_j S_j b_{i,j} f_j^n$$

and therefore

$$\sum_{i=0}^{I^h} \Delta x_i f_i^{n+1} \leq (1 + R \|bS\|_{L^\infty} \Delta t) \sum_{i=0}^{I^h} \Delta x_i f_i^n.$$

Finally, using (20) at step n and the inequality $1 + x < \exp(x)$ for all $x > 0$ gives

$$\sum_{i=0}^{I^h} \Delta x_i f_i^{n+1} \leq \|f^{in}\|_{L^1} e^{R \|bS\|_{L^\infty} t^{n+1}}$$

and therefore the result (19) follows. □

Next we will prove the equiintegrability for the function f^h . The following property on convex functions, as stated in the La Vallée Poussin theorem [8, Proposition I.1.1], and Lemma 4.4 are used to show this result. Since $f^{in} \in L^1]0, R]$, hence by the La Vallée Poussin theorem, there exists a convex function $\Phi \geq 0$, continuously differentiable on $\mathbb{R}_{>0}$ with $\Phi(0) = 0$, $\Phi'(0) = 1$ such that Φ' is concave,

$$\frac{\Phi(r)}{r} \rightarrow \infty, \text{ as } r \rightarrow \infty$$

and

$$(23) \quad \int_0^R \Phi(f^{in})(x) dx < +\infty.$$

Lemma 4.4. [12, Lemma B.1.] *Let $\Phi \in C^1(\mathbb{R}_{>0})$ be convex such that Φ' is concave, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(r)/r \rightarrow \infty$ as $r \rightarrow \infty$. Then for all $(x, y) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,*

$$x\Phi'(y) \leq \Phi(x) + \Phi(y).$$

Now, we are in a position to prove the equiintegrability in the following.

Proposition 4.5. *Let $f^{in} \geq 0 \in L^1]0, R]$ and let f^h be defined for all h and Δt by (8) where Δt satisfies (14). Then the family $(f^h)_{(h, \Delta t)}$ is weakly relatively sequentially compact in $L^1]0, T[\times]0, R]$.*

Proof. Our aim is to get a similar estimate as (23) for the function f^h . We know that the integral of $\Phi(f^h)$ is related to the sequence f_i^n through

$$\begin{aligned} \int_0^T \int_0^R \Phi(f^h(t, x)) dx dt &= \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \int_{\tau_n} \int_{\Lambda_i^h} \Phi \left(\sum_{k=0}^{N-1} \sum_{j=0}^{I^h} f_j^k \chi_{\Lambda_j^h}(x) \chi_{\tau_k}(t) \right) dx dt \\ &= \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \Delta t \Delta x_i \Phi(f_i^n). \end{aligned}$$

Since Φ is a convex function, we can estimate

$$(f_i^{n+1} - f_i^n) \Phi'(f_i^{n+1}) \geq \Phi(f_i^{n+1}) - \Phi(f_i^n).$$

Hence, multiplying this equation by Δx_i and taking summation over i on both sides we get

$$\sum_{i=0}^{I^h} \Delta x_i [\Phi(f_i^{n+1}) - \Phi(f_i^n)] \leq \sum_{i=0}^{I^h} \Delta x_i [(f_i^{n+1} - f_i^n) \Phi'(f_i^{n+1})].$$

By using the discrete equation (8), it can be rewritten as

$$\sum_{i=0}^{I^h} \Delta x_i [\Phi(f_i^{n+1}) - \Phi(f_i^n)] \leq \sum_{i=0}^{I^h} \frac{\Delta t}{x_i} (\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n) \Phi'(f_i^{n+1}).$$

Since Φ is a convex function, its derivative is non-decreasing. Therefore, $\Phi'(0) = 1$ implies that $\Phi'(x) > 0$ for $x \geq 0$. Further simplification as in (22) gives

$$\begin{aligned} \sum_{i=0}^{I^h} \Delta x_i [\Phi(f_i^{n+1}) - \Phi(f_i^n)] &\leq \sum_{i=0}^{I^h} \frac{\Delta t}{x_i} \sum_{j=i+1}^{I^h} x_i S_j b_{i,j} f_j^n \Delta x_j \Delta x_i \Phi'(f_i^{n+1}) \\ &\leq \|bS\|_{L^\infty} \Delta t \sum_{i=0}^{I^h} \sum_{j=i+1}^{I^h} \Delta x_j \Delta x_i f_j^n \Phi'(f_i^{n+1}). \end{aligned}$$

Using the property $x\Phi'(y) \leq \Phi(x) + \Phi(y)$ from Lemma 4.4, it reduces to

$$\begin{aligned} \sum_{i=0}^{I^h} \Delta x_i [\Phi(f_i^{n+1}) - \Phi(f_i^n)] &\leq \|bS\|_{L^\infty} \Delta t \sum_{i=0}^{I^h} \Delta x_i \sum_{j=i+1}^{I^h} \Delta x_j (\Phi(f_j^n) + \Phi(f_i^{n+1})) \\ &\leq (\|bS\|_{L^\infty} \Delta t R) \left(\sum_{j=0}^{I^h} \Delta x_j \Phi(f_j^n) + \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^{n+1}) \right). \end{aligned}$$

Changing the index from j to i for the first term on the right-hand side and taking $\|bS\|_{L^\infty} R = C(T, R)$, we obtain

$$(1 - \Delta t C(T, R)) \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^{n+1}) \leq (1 + \Delta t C(T, R)) \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^n).$$

Equivalently, it can be rewritten as

$$(24) \quad (1 - \Delta t C(T, R)) \sum_{i=0}^{I^h} \Delta x_i (\Phi(f_i^{n+1}) - \Phi(f_i^n)) \leq 2\Delta t C(T, R) \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^n).$$

This gives using $\lambda = \frac{2C(T,R)}{1-\Delta t C(T,R)} > 0$

$$\sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^{n+1}) \leq (1 + \lambda \Delta t) \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^n)$$

for any n . Hence, we achieve the result that

$$\begin{aligned} \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^n) &\leq (1 + \lambda \Delta t)^n \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^0) \\ &\leq \exp(\lambda \Delta t n) \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^0). \end{aligned}$$

For time $t \in \tau_n = [t_n, t_{n+1}[$ the above expression becomes

$$\begin{aligned} \int_0^R \Phi(f^h(t, x)) dx &\leq \exp(\lambda t) \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^{in}) \\ &\leq \exp(\lambda t) \sum_{i=0}^{I^h} \Delta x_i \Phi \left(\frac{1}{\Delta x_i} \int_{\Lambda_i^h} f^{in}(x) dx \right). \end{aligned}$$

We apply Jensen’s inequality to get

$$\int_0^R \Phi(f^h(t, x)) dx \leq \exp(\lambda t) \sum_{i=0}^{I^h} \Delta x_i / \Delta x_i \int_{\Lambda_i^h} \Phi(f^{in}(x)) dx.$$

Equivalently, we have

$$\int_0^R \Phi(f^h(t, x)) dx \leq \exp \left(\frac{2C(T, R)t}{1 - \Delta t C(T, R)} \right) \int_0^R \Phi(f^{in}(x)) dx.$$

As we know from (14) that $1 - \Delta t C(T, R) \geq 1 - \theta$. This implies that

$$\int_0^R \Phi(f^h(t, x)) dx \leq \exp \left(\frac{2C(T, R)t}{1 - \theta} \right) \int_0^R \Phi(f^{in}(x)) dx, \quad \text{for all } t \in [0, T[$$

and it concludes the proof. □

Hence, the sequence $(f^h)_{h \in \mathbb{N}}$ is weakly compact in L^1 due to the Dunford-Pettis theorem. Here, the exponent is uniformly bounded with respect to h and Δt as long as the time step restriction (14) holds. This implies that there exists a subsequence of $(f^h)_{h \in \mathbb{N}}$ and a function $f \in L^1(]0, T[\times]0, R])$ such that $f^h \rightharpoonup f$ as $h \rightarrow 0$.

So far we have seen that the sequence f_i^n is built from the numerical scheme as a sequence of step functions f^h depending on the mesh size h and the time step Δt . We have already seen the weak compactness of this sequence. Now in order to prove Theorem 4.1, it remains to show that the discrete breakage flux converges weakly towards the continuous flux when it is written in terms of the function f^h . This is done in Lemma 4.7 later.

We use the following point approximations for further analysis. First we define the midpoint approximation as

$$X^h : x \in]0, R[\rightarrow X^h(x) = \sum_{i=0}^{I^h} x_i \chi_{\Lambda_i^h}(x).$$

Then right and left endpoint approximations are taken respectively as

$$\Xi^h : x \in]0, R[\rightarrow \Xi^h(x) = \sum_{i=0}^{I^h} x_{i+1/2} \chi_{\Lambda_i^h}(x),$$

and

$$\xi^h : x \in]0, R[\rightarrow \xi^h(x) = \sum_{i=0}^{I^h} x_{i-1/2} \chi_{\Lambda_i^h}(x).$$

It should be mentioned that the approximations $(X^h)_h$, $(\Xi^h)_h$ and $(\xi^h)_h$ converge pointwise, i.e. for all $x \in]0, R[$,

$$X^h(x) \rightarrow x, \quad \Xi^h(x) \rightarrow x \quad \text{and} \quad \xi^h(x) \rightarrow x$$

as $h \rightarrow 0$. We also use the following classical lemma to prove the convergence of the numerical flux towards the continuous flux. The proof of this lemma is based on the Dunford-Pettis and Egorov theorems.

Lemma 4.6. [12, Lemma A.2] *Let Ω be an open subset of \mathbb{R}^m and let there exist a constant $k > 0$ and two sequences $(v_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ such that $(v_n) \in L^1(\Omega)$, $v \in L^1(\Omega)$ and*

$$\begin{aligned} &v_n \rightharpoonup v, \quad \text{weakly in } L^1(\Omega), \text{ as } n \rightarrow \infty, \\ &(w_n) \in L^\infty(\Omega), w \in L^\infty(\Omega), \text{ and for all } n \in \mathbb{N}, |w_n| \leq k \text{ with} \\ &w_n \rightarrow w, \quad \text{almost everywhere (a.e.) in } \Omega, \text{ as } n \rightarrow \infty. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} \|v_n(w_n - w)\|_{L^1(\Omega)} = 0$$

and

$$v_n w_n \rightharpoonup v w, \quad \text{weakly in } L^1(\Omega), \text{ as } n \rightarrow \infty.$$

Consider the definitions of f^h , b^h and S^h given by (11), (12) and (13) respectively. The following lemma state the convergence result of the numerical flux towards the continuous flux.

Lemma 4.7. *Let us define the approximation of the fragmentation terms as*

$$\mathcal{F}^h(t, x) = \int_0^R \int_0^R \chi_{[0, \Xi^h(x)]}(u) \chi_{[\Xi^h(x), R]}(v) X^h(u) b^h(u, v) S^h(v) f^h(t, v) dudv.$$

There exists a subsequence of $(f^h)_{h \in \mathbb{N}}$, such that

$$\mathcal{F}^h \rightharpoonup \mathcal{F}_c^R \text{ in } L^1(]0, T[\times]0, R]) \text{ as } h \rightarrow 0.$$

Before proving this lemma, it is worth to mention that actually the $\mathcal{F}^h(t, x)$ coincide with \mathcal{F}_i^n whenever $t \in \tau_n$ and $x \in \Lambda_i^h$. It can be seen easily that for $x \in \Lambda_i^h$

$$\begin{aligned} \mathcal{F}^h(t, x) &= \int_{x_{i+1/2}}^R \int_0^{x_{i+1/2}} X^h(u) b^h(u, v) S^h(v) f^h(t, v) dudv \\ &= \sum_{j=i+1}^{I^h} \int_{\Lambda_j^h} \sum_{k=0}^i \int_{\Lambda_k^h} \left[X^h(u) \left(\sum_{\ell=0}^{I^h} \sum_{m=0}^{I^h} b_{m, \ell} \chi_{\Lambda_m^h}(u) \chi_{\Lambda_\ell^h}(v) \right) \left(\sum_{\ell=0}^{I^h} S_\ell \chi_{\Lambda_\ell^h}(v) \right) \right. \\ &\quad \cdot \left. \left(\sum_{\ell=0}^{I^h} f_\ell^n \chi_{\Lambda_\ell^h}(v) \right) \right] dudv \\ &= \sum_{j=i+1}^{I^h} \sum_{k=0}^i \int_{\Lambda_j^h} \int_{\Lambda_k^h} x_k b_{k, j} S_j f_j^n dudv = \mathcal{F}_{i+1/2}^n. \end{aligned}$$

Proof. [Lemma 4.7]

We know that for all $(t, x) \in]0, T[\times]0, R[$ and for $u \in]0, R[$ almost everywhere that the sequence

$$X^h(\cdot)b^h(\cdot, v)S^h(v) \in L^\infty]0, R[\quad \text{for almost all } v \in]0, R[.$$

It is uniformly bounded and

$$\chi_{[0, \Xi^h(x)]}(u)\chi_{[\Xi^h(x), R]}(v)X^h(u)b^h(u, v)S^h(v) \rightarrow \chi_{[0, x]}(u)\chi_{[x, R]}(v)ub(u, v)S(v)$$

pointwise almost everywhere as $h \rightarrow 0$. We also know that $f^h \rightharpoonup f$ in $L^1]0, R[$. Hence, applying Lemma 4.6 yields

$$\chi_{[0, \Xi^h(x)]}(u)\chi_{[\Xi^h(x), R]}(v)X^h(u)b^h(u, v)S^h(v)f^h(t, v) \rightharpoonup \chi_{[0, x]}(u)\chi_{[x, R]}(v)ub(u, v)S(v)f(t, v)$$

in $L^1]0, R[$. Therefore, we have

$$(25) \quad \int_0^R \chi_{[0, \Xi^h(x)]}(u)\chi_{[\Xi^h(x), R]}(v)X^h(u)b^h(u, v)S^h(v)f^h(t, v)dv \rightarrow \int_0^R \chi_{[0, x]}(u)\chi_{[x, R]}(v)ub(u, v)S(v)f(t, v)dv.$$

This implies that (25) holds for each t, x and almost every u . Finally, by applying dominated convergence theorem we get

$$\mathcal{F}^h(t, x) \rightarrow \mathcal{F}_c^R(t, x)$$

for every $(t, x) \in]0, T[\times]0, R[$. As \mathcal{F}^h is bounded, this pointwise convergence implies weak convergence for \mathcal{F}^h . \square

Now we have gathered all the results needed to prove Theorem 4.1. The proof is given below. For this, let us consider a test function $\varphi \in C^1([0, T[\times [0, R])$ which is compactly supported. For Δt small enough, the support of φ with respect to t satisfies $\text{Supp}_t \varphi \subset [0, t_{N-1}]$. Define the finite volume (in time) and left endpoint (in space) approximation of φ on $\tau_n \times \Lambda_i^h$ by

$$\varphi_i^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \varphi(t, x_{i-1/2}) dt.$$

Multiplying (8) by φ_i^n and summing over $n \in \{0, \dots, N-1\}$ as well as $i \in \{0, \dots, I^h\}$ give

$$\sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \left[\Delta x_i x_i (f_i^{n+1} - f_i^n) \varphi_i^n - \Delta t \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right) \varphi_i^n \right] = 0.$$

If we open the summation for both i and n , discrete integration by parts yields

$$(26) \quad \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \Delta x_i x_i f_i^{n+1} (\varphi_i^{n+1} - \varphi_i^n) + \sum_{i=0}^{I^h} \Delta x_i x_i f_i^{i n} \varphi_i^0 - \sum_{n=0}^{N-1} \sum_{i=0}^{I^h-1} \Delta t \mathcal{F}_{i+1/2}^n (\varphi_{i+1}^n - \varphi_i^n) = 0.$$

Now, we evaluate the first two terms on the left-hand side by writing them in terms of the function f^h as

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \Delta x_i x_i f_i^{n+1} (\varphi_i^{n+1} - \varphi_i^n) + \sum_{i=0}^{I^h} \Delta x_i x_i f_i^{in} \varphi_i^0 = \\ & \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \int_{\tau_{n+1}} \int_{\Lambda_i^h} X^h(x) f^h(t, x) \frac{\varphi(t, \xi^h(x)) - \varphi(t - \Delta t, \xi^h(x))}{\Delta t} dx dt \\ & + \sum_{i=0}^{I^h} \int_{\Lambda_i^h} X^h(x) f^h(0, x) \frac{1}{\Delta t} \int_0^{\Delta t} \varphi(t, \xi^h(x)) dt dx. \end{aligned}$$

Further it can be written as

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \Delta x_i x_i f_i^{n+1} (\varphi_i^{n+1} - \varphi_i^n) + \sum_{i=0}^{I^h} \Delta x_i x_i f_i^{in} \varphi_i^0 = \\ & \int_{\Delta t}^T \int_0^R X^h(x) f^h(t, x) \frac{\varphi(t, \xi^h(x)) - \varphi(t - \Delta t, \xi^h(x))}{\Delta t} dx dt \\ & + \int_0^R X^h(x) f^h(0, x) \frac{1}{\Delta t} \int_0^{\Delta t} \varphi(t, \xi^h(x)) dt dx. \end{aligned}$$

Since, $\varphi \in C^1([0, T] \times [0, R])$ with compact support and the derivative of φ is bounded, we have

$$\frac{1}{\Delta t} \int_0^{\Delta t} \varphi(t, \xi^h(x)) dt \rightarrow \varphi(0, x)$$

uniformly with respect to t, x as $\max\{h, \Delta t\}$ goes to 0. Moreover, we know that $X^h(x)$ converges pointwise in $[0, R]$ and $f^h(0, x) \rightarrow f^{in}$ in $L^1[0, R]$. Thus we achieve by using Lemma 4.6

$$\int_0^R X^h(x) f^h(0, x) \frac{1}{\Delta t} \int_0^{\Delta t} \varphi(t, \xi^h(x)) dt dx \rightarrow \int_0^R x f^{in}(x) \varphi(0, x) dx$$

as $\max\{h, \Delta t\}$ goes to 0.

Now, using Taylor expansion of the smooth function φ yields

$$\begin{aligned} & \frac{\varphi(t, \xi^h(x)) - \varphi(t - \Delta t, \xi^h(x))}{\Delta t} \\ & = \frac{\varphi(t, x) + (x - \xi^h(x)) \frac{\partial \varphi}{\partial x} - \varphi(t, x) + \Delta t \frac{\partial \varphi}{\partial t} - (x - \xi^h(x)) \frac{\partial \varphi}{\partial x} + O(h \Delta t)}{\Delta t}. \end{aligned}$$

It implies that

$$\frac{\varphi(t, \xi^h(x)) - \varphi(t - \Delta t, \xi^h(x))}{\Delta t} \rightarrow \frac{\partial \varphi}{\partial t}(t, x)$$

uniformly as $\max\{h, \Delta t\}$ goes to 0. Applying Lemma 4.6, together with Proposition 4.5, ensures that for $\max\{h, \Delta t\}$ goes to 0

$$\int_0^T \int_0^R X^h(x) f^h(t, x) \frac{\varphi(t, \xi^h(x)) - \varphi(t - \Delta t, \xi^h(x))}{\Delta t} dx dt \rightarrow \int_0^T \int_0^R x f(t, x) \frac{\partial \varphi}{\partial t}(t, x) dx dt.$$

Hence, we obtain

$$\begin{aligned} & \int_{\Delta t}^T \int_0^R X^h(x) f^h(t, x) \underbrace{\frac{\varphi(t, \xi^h(x)) - \varphi(t - \Delta t, \xi^h(x))}{\Delta t}}_A dx dt \\ &= \int_0^T \int_0^R A dx dt - \int_0^{\Delta t} \int_0^R A dx dt \rightarrow \int_0^T \int_0^R x f(t, x) \frac{\partial \varphi}{\partial t}(t, x) dx dt \end{aligned}$$

as $\max\{h, \Delta t\} \rightarrow 0$. Finally, writing the remaining third term of the equation (26) in terms of \mathcal{F}^h gives

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{i=0}^{I^h-1} \Delta t \mathcal{F}_{i+1/2}^n (\varphi_{i+1}^n - \varphi_i^n) \\ &= \sum_{n=0}^{N-1} \sum_{i=0}^{I^h-1} \int_{\tau_n} \int_{\Lambda_i^h} \mathcal{F}_{i+1/2}^n \frac{1}{\Delta x_i} [\varphi(t, x_{i+1/2}) - \varphi(t, x_{i-1/2})] dx dt \\ &= \int_0^T \int_0^{R-\Delta x_1 h} \mathcal{F}^h(t, x) \frac{\partial \varphi}{\partial x}(t, x) dx dt. \end{aligned}$$

By using the weak convergence for the flux from Lemma 4.7, i.e. $\mathcal{F}^h \rightharpoonup \mathcal{F}_c^R$ in $L^1([0, T] \times]0, R])$, we determine

$$\begin{aligned} \int_0^T \int_0^{R-\Delta x_1 h} \mathcal{F}^h(t, x) \frac{\partial \varphi}{\partial x}(t, x) dx dt &= \left(\int_0^T \int_0^R - \int_0^T \int_{\Delta x_1 h} \right) \mathcal{F}^h(t, x) \frac{\partial \varphi}{\partial x}(t, x) dx dt \\ &\rightarrow \int_0^T \int_0^R \mathcal{F}_c^R \frac{\partial \varphi}{\partial x}(t, x) dx dt, \quad \text{as } h \rightarrow 0. \end{aligned}$$

Therefore, the corresponding terms in (16) are obtained.

5. Conclusions

In this article a mass conservative formulation of the multiple breakage PBE was considered. We then demonstrated the convergence of finite volume approximations towards a weak solution to the continuous multiple breakage equations. This investigation was done in L^1 space by using the Dunford-Pettis and La Vallée Poussin theorem which required to show the equiboundedness and equiintegrability of the numerical solution. The analysis was performed by assuming certain growth condition, i.e. locally bounded, on the product of breakage and selection functions. The stability was also discussed under some CFL condition on time step. In the future it would be interesting to see how one can enlarge the class of breakage kernels.

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