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On the Sum of Operators of p-Laplacian Types

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Abstract. The goal of this paper is to study operators sum of p-Laplacian type operators. We address the problems of existence and uniqueness of solutions, this last point leading to some challenging issues in the case of quasilinear combinations of such p-Laplacians.

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1 Introduction and notation

We will denote by Ω a bounded open subset of $\mathbb{R}^n, n \geq 1$. Let us consider $p_1, p_2, \ldots, p_N$ real numbers such that

$$1 < p_1 < p_2 < \ldots < p_N,$$

and $a_i(x,u), i=1,\ldots,N$, Carathéodory functions, i.e. such that for every i,

$$\begin{aligned} x &\rightarrow a_i(x,u) \quad \text{is measurable}, \quad \forall u \in \mathbb{R}, \\ u &\rightarrow a_i(x,u) \quad \text{is continuous a.e. } x \in \Omega. \end{aligned}$$

We will suppose that for some positive constants λ, Λ,

$$\begin{aligned} &0 \leq a_i(x,u) \leq \Lambda, \quad \forall i = 1,\ldots,N-1, \\ &\lambda \leq a_N(x,u) \leq \Lambda, \quad \forall u \in \mathbb{R} \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{1.1}$$

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We would like to consider problems of the following type:

$$\begin{cases} u \in W_0^{1,p_N}(\Omega), \\ -\nabla\cdot\left(\sum_{i=1}^{N} a_i(x,u)|\nabla u|^{p_i-2}\nabla u\right) = f \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

or under the weak form

$$\begin{cases} u \in W_0^{1,p_N}(\Omega), \\ \int_\Omega \sum_{i=1}^{N} a_i(x,u)|\nabla u|^{p_i-2}\nabla u\cdot\nabla v dx = \langle f,v\rangle, \quad \forall v \in W_0^{1,p_N}(\Omega), \end{cases} \tag{1.3}$$

where $W_0^{1,p}(\Omega)$ denotes the usual Sobolev space of functions in $L^p(\Omega)$ with derivatives in $L^p(\Omega)$, vanishing on the boundary of Ω, $f \in W^{-1,p_N'}(\Omega)$ the dual space of $W_0^{1,p_N}(\Omega)$ (cf. [5]). Recall that for $p\in\mathbb{R}, p>1, p'$ denotes the conjugate of p given by $p'=p/(p-1)$.

We suppose that $W_0^{1,p}(\Omega)$-spaces are equipped with the norm

$$\|\nabla v\|_p = \left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}},$$

and their duals $W^{-1,p'}(\Omega)$ with the strong dual norm defined as

$$|f|_* = \sup_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{|\langle f,v\rangle|}{\|\nabla v\|_p}.$$

Such operators appeared some decades ago in particular as Euler equation of problems of calculus of variations (cf. [7], [8], [10]), the idea being to consider energy functionals presenting at the same time different growth and to analyse the regularity of the possible minimisers (see [9], which contains many interesting references, and also [6]). Later (cf. [4], [11]) problems of this type were supposed to model situations where different phases coexist, two in general, leading to the notion of (p,q)-Laplacian. Of course here we consider the sum of several pseudo p-Laplacians and the Eq. (1.3) is not the Euler equation of some energy except perhaps in the case when the a_i's are constant. We do not pretend either having in mind applications. We are more guided by the challenges offered by this kind of problems when existence and uniqueness of solution are concerned.

In the next section we develop a theory of existence of solution based on the theory of monotone operators. The subsequent part addresses different issues of uniqueness or non uniqueness. In dimension one we are able to construct some $a_1(x,u)=a(x,u)$ leading to non uniqueness in the case $N=1$ and to prove uniqueness when the a_i's are say continuous and Lipschitz continuous in u. In higher dimensions one has to restrict ourselves to special a_i's or to a single operator but the results that we are able to show do not rely on Lipschitz continuity.

2 Existence result

Let us first prove the following existence result.

Theorem 2.1. *We assume that the $a_i(x,u)$ are Carathéodory functions satisfying* (1.1). *If $f\in W^{-1,p'_N}(\Omega)$, there exists a solution u to* (1.3).

Proof. Let $w\in L^{p_N}(\Omega)$. We claim that there exists a unique $u=S(w)$ solution to

$$\begin{cases} u\in W_0^{1,p_N}(\Omega), \\ \displaystyle\int_\Omega \sum_{i=1}^N a_i(x,w)|\nabla u|^{p_i-2}\nabla u\cdot\nabla v dx=\langle f,v\rangle, \quad \forall v\in W_0^{1,p_N}(\Omega). \end{cases} \tag{2.1}$$

Note that the operator

$$-\nabla\cdot\left(\sum_{i=1}^N a_i(x,w)|\nabla u|^{p_i-2}\nabla u\right)$$

is monotone, hemicontinuous, coercive from $W_0^{1,p_N}(\Omega)$ into its dual since

$$a_i(x,w)|\nabla u|^{p_i-2}\nabla u\in L^{p'_i}\subset L^{p'_N}(\Omega).$$

Indeed, $p_i<p_N$ implies $p'_i>p'_N$. The coerciveness of the operator is insured by (1.1), cf. the inequality just below. We will be done if we can show that the mapping S has a fixed point. First, taking $v=u$ in (2.1) we deduce

$$\begin{aligned}\lambda\int_\Omega|\nabla u|^{p_N}dx &\le \int_\Omega\sum_{i=1}^N a_i(x,w)|\nabla u|^{p_i-2}\nabla u\cdot\nabla u dx=\langle f,u\rangle \\ &\le |f|_*\|\nabla u\|_{p_N}.\end{aligned}$$

Thus, it comes

$$\|\nabla u\|_{p_N}\le\left(\frac{|f|_*}{\lambda}\right)^{\frac{1}{p_N-1}}.$$

We denote by C_{p_N} the constant in the Poincaré inequality (cf. [5]) such that

$$|u|_{p_N} \le C_{p_N} \|\nabla u\|_{p_N}, \quad \forall u \in W_0^{1,p_N}(\Omega),$$

where $|u|_p$ is the $L^p(\Omega)$-norm of u. Then we have

$$|u|_{p_N} \le C_{p_N} \|\nabla u\|_{p_N} \le C_{p_N} \left(\frac{|f|_*}{\lambda} \right)^{\frac{1}{p_N - 1}} = K. \tag{2.2}$$

Thus, the mapping S goes from the ball

$$B = \left\{ u \in L^{p_N}(\Omega) : |u|_{p_N} \le K \right\}$$

into itself and is relatively compact thanks to the estimate above. We will be done, by the Schauder fixed point theorem, if we show that S is continuous from B into B. For that consider a sequence w_n such that $w_n \to w$ in $L^{p_N}(\Omega)$. Without loss of generality we can assume that $w_n \to w$ a.e. in Ω. Set $u_n = S(w_n)$. From (2.2) it follows that u_n is bounded in $W_0^{1,p_N}(\Omega)$ and up to a subsequence there exists $u \in W_0^{1,p_N}(\Omega)$ such that

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u \quad \text{in } L^{p_i}(\Omega), \quad \forall i, \\ w_n &\to w \qquad \text{a.e. in } \Omega. \end{aligned}$$

We know that u_n satisfies

$$\int_\Omega \sum_{i=1}^N a_i(x, w_n) |\nabla u_n|^{p_i - 2} \nabla u_n \cdot \nabla (v - u_n) dx = \langle f, v - u_n \rangle, \quad \forall v \in W_0^{1,p_N}(\Omega)$$

and by monotonicity of the operators,

$$\int_\Omega \sum_{i=1}^N a_i(x, w_n) |\nabla v|^{p_i - 2} \nabla v \cdot \nabla (v - u_n) dx \ge \langle f, v - u_n \rangle, \quad \forall v \in W_0^{1,p_N}(\Omega). \tag{2.3}$$

By the Lebesgue theorem one has

$$a_i(x, w_n) |\nabla v|^{p_i - 2} \nabla v \;\to\; a_i(x, w) |\nabla v|^{p_i - 2} \nabla v \quad \text{in } L^{p_i'}(\Omega).$$

Passing to the limit in (2.3), we get

$$\int_\Omega \sum_{i=1}^N a_i(x, w) |\nabla v|^{p_i - 2} \nabla v \cdot \nabla (v - u) dx \ge \langle f, v - u \rangle, \quad \forall v \in W_0^{1,p_N}(\Omega). \tag{2.4}$$

Replacing v by $u \pm \delta v$ we obtain

$$\int_\Omega \sum_{i=1}^N a_i(x,w)|\nabla u \pm \delta v|^{p_i-2}\nabla(u\pm\delta v)\cdot\nabla(\pm\delta v)dx \geq \langle f,\pm\delta v\rangle, \quad \forall v\in W_0^{1,p_N}(\Omega),$$

i.e.

$$\int_\Omega \sum_{i=1}^N a_i(x,w)|\nabla u \pm \delta v|^{p_i-2}\nabla(u\pm\delta v)\cdot\nabla(\pm v)dx \geq \langle f,\pm v\rangle, \quad \forall v\in W_0^{1,p_N}(\Omega).$$

Letting $\delta \to 0$, we obtain

$$\int_\Omega \sum_{i=1}^N a_i(x,w)|\nabla u|^{p_i-2}\nabla u\cdot\nabla v dx = \langle f,v\rangle, \quad \forall v\in W_0^{1,p_N}(\Omega),$$

and thus $u = Sw$. Note that the whole sequence u_n converges toward u since the limit is unique. This completes the proof of existence of a solution to (1.3). □

3 Uniqueness issues

We suppose here that we are in dimension 1 with $\Omega = (\eta_1,\eta_2)$.

Theorem 3.1. *One can construct a continuous function $a(x,u)$ such the problem*

$$\begin{cases} u\in W_0^{1,p}(\Omega), \\ \int_\Omega a(x,u)|u'|^{p-2}u'v'dx = \langle f,v\rangle, \quad \forall v\in W_0^{1,p}(\Omega) \end{cases} \tag{3.1}$$

admits several solutions.

Proof. We use a construction similar to one in [1]. Set

$$u(x) = (x-\eta_1)(\eta_2 - x), \quad f = -(|u'|^{p-2}u')'.$$

One has clearly

$$\begin{cases} u\in W_0^{1,p}(\Omega), \\ \int_\Omega |u'|^{p-2}u'v'dx = \langle f,v\rangle, \quad \forall v\in W_0^{1,p}(\Omega). \end{cases} \tag{3.2}$$

Let ω be a nondecreasing, continuous function such that

$$\omega(0)=0, \quad \omega(t)>0, \quad \forall t>0, \quad \int_{0^+} \frac{ds}{\omega(s)}<+\infty, \tag{3.3}$$

$$\frac{\omega(t)}{t} \quad \text{is non increasing} \tag{3.4}$$

($t^\alpha, \alpha<1$ would be suitable). Set

$$\theta(s)=\int_0^s \frac{dt}{\omega(t)}. \tag{3.5}$$

θ is one-to-one mapping from $[0,T]$ into $[0,\theta(T)]$ for every $T>0$. Let us denote by θ^{-1} its inverse. One has

$$\frac{d}{dy}\theta^{-1}(y)=\omega\big(\theta^{-1}(y)\big). \tag{3.6}$$

Then we define

$$v(x)=\begin{cases} u(x)+\theta^{-1}(x-\eta_1) & \text{in a neighbourhood of } \eta_1, \\ u(x)+\theta^{-1}(\eta_2-x) & \text{in a neighbourhood of } \eta_2, \end{cases} \tag{3.7}$$

and we assume that

$$v>u, \quad \frac{u'}{v'}>0 \quad \text{on } (\eta_1,\eta_2). \tag{3.8}$$

To fulfil the second condition it is enough to have v increasing on $(\eta_1,(\eta_1+\eta_2)/2)$, decreasing on $((\eta_1+\eta_2)/2,\eta_2)$, $v''((\eta_1+\eta_2)/2)<0$ since, as $v'((\eta_1+\eta_2)/2)=0$,

$$\lim_{x\to\frac{\eta_1+\eta_2}{2}} \frac{u'}{v'}(x)=(u''v'')\left(\frac{\eta_1+\eta_2}{2}\right).$$

It is clear that it is always possible to find such a v. Then for $x,u\in\mathbb{R}$ we define $a(x,u)$ as

$$a(x,u)=\begin{cases} 1, & \text{if} \quad x\notin(\eta_1,\eta_2), \\ 1, & \text{if} \quad u\le u(x), \quad x\in(\eta_1,\eta_2), \\ \left(\dfrac{u'}{v'}\right)^{p-1}, & \text{if} \quad u\ge v(x), \quad x\in(\eta_1,\eta_2), \\ \delta+(1-\delta)\left(\dfrac{u'}{v'}\right)^{p-1}, & \text{if} \quad u=\delta u(x)+(1-\delta)v(x), \quad x\in(\eta_1,\eta_2). \end{cases} \tag{3.9}$$

Clearly $a(x,u)$ is continuous on $\mathbb{R}^2$. Note that $u'(\eta_1)=v'(\eta_1)$ and $u'(\eta_2)=v'(\eta_2)$. Now it is not Lipschitz continuous in u. Indeed, let us denote by $\omega_a(t)$ the modulus of continuity of $a(x,u)$ with respect to u, namely

$$\omega_a(t)=\sup_{x\in\Omega,|u-v|\le t}|a(x,u)-a(x,v)|.$$

For t small there exists x near η_1 such that $v(x)-u(x)=t$. Moreover, one has

$$\begin{aligned} a\big(x,u(x)\big)-a\big(x,v(x)\big)&=1-\left(\frac{u'(x)}{v'(x)}\right)^{p-1}\\ &=\frac{1}{v'(x)^{p-1}}\big(v'(x)^{p-1}-u'(x)^{p-1}\big). \end{aligned}$$

Recall that for x close to η_1,

$$(v-u)'(x)=\frac{d}{dx}\theta^{-1}(x-\eta_1)=\omega\big(\theta^{-1}(x-\eta_1)\big)=\omega\big(v(x)-u(x)\big).$$

This implies that $v'(x)>u'(x)$ for x close to η_1. We also have

$$\begin{aligned} &\big(v'(x)^{p-1}-u'(x)^{p-1}\big)\\ &=\int_0^1\frac{d}{ds}\big\{u'(x)+s\big(v'(x)-u'(x)\big)\big\}^{p-1}ds\\ &=\int_0^1(p-1)\big\{u'(x)+s\big(v'(x)-u'(x)\big)\big\}^{p-2}ds(v-u)'(x)\\ &=\omega(t)\int_0^1(p-1)\big\{u'(x)+s\big(v'(x)-u'(x)\big)\big\}^{p-2}ds. \end{aligned}$$

Clearly $v'(x)$, $u'(x)$ are bounded and bounded away from 0 near η_1. Thus

$$\begin{aligned} &a\big(x,u(x)\big)-a\big(x,v(x)\big)\\ &=\frac{1}{v'(x)^{p-1}}\int_0^1(p-1)\big\{u'(x)+s\big(v'(x)-u'(x)\big)\big\}^{p-2}ds\,\omega(t), \end{aligned}$$

and

$$\omega_a(t)=\sup_{x\in\Omega,|u-v|\le t}|a(x,u)-a(x,v)|\ge C\omega(t)$$

for some constant C. This implies that

$$\int_{0^+}\frac{ds}{\omega_a(s)}\le\int_{0^+}\frac{ds}{\omega(s)}<+\infty,$$

which is impossible if $\omega_a(t) \sim Kt$. This shows that $a(x,u)$ is not Lipschitz continuous in u. Now one has

$$\begin{aligned} a\big(x,v(x)\big)|v'|^{p-2}v' &= \left(\frac{|u'|}{|v'|}\right)^{p-1}|v'|^{p-2}v' = |u'|^{p-1}\frac{v'}{|v'|} \\ &= |u'|^{p-1}\frac{u'}{|u'|} = |u'|^{p-2}u', \\ a\big(x,u(x)\big)|u'|^{p-2}u' &= |u'|^{p-2}u', \end{aligned}$$

since $v'/|v'| = u'/|u'|$. Thus, both u and v are solution to (3.1). This completes the proof of the theorem. □

We study now a particular example in dimension 1 where, on the contrary, we are able to prove uniqueness of solution. For that let us consider a function f defined on $\Omega = (\eta_1, \eta_2)$ and satisfying

$$f \in L^1(\Omega). \tag{3.10}$$

Note that in one dimension $L^1(\Omega) \subset W^{-1,p'_N}(\Omega)$ since $W_0^{1,p_N}(\Omega) \subset L^\infty(\Omega)$ (see, for instance [2]).

For $i=1,\dots,N$ let $a_i(x,u)$ be continuous functions satisfying (1.1). Suppose that for

$$1 < p_1 < p_2 < \dots < p_N, \tag{3.11}$$

u is weak solution to

$$\begin{aligned} &-\left(\sum_{i=1}^N a_i\big(x,u(x)\big)|u'|^{p_i-2}u'\right)' = f \quad \text{in } \Omega, \\ &u(\eta_1) = u(\eta_2) = 0. \end{aligned} \tag{3.12}$$

Let us first establish a lemma which will be useful in what follows to consider u as solution of a Cauchy problem.

Lemma 3.1. *Let us denote by a_i, $i=1,\dots,N$ positive constants. For $a=(a_1,\dots,a_N)$ we denote by $F_a(z)$ the inverse function of the increasing function from $\mathbb{R}$ into $\mathbb{R}$*

$$X \rightarrow \sum_{i=1}^N a_i |X|^{p_i-2}X.$$

Then one has for some constant c_{p_1}, see (3.16),

$$|F_a(z)-F_{a'}(z)| \leq \frac{1}{c_{p_1}a_1}\sum_{i=1}^{N}|a_i'-a_i|\left\{\left(\frac{|z|}{a_1}\right)^{\frac{1}{p_1-1}}+\left(\frac{|z|}{a_1'}\right)^{\frac{1}{p_1-1}}\right\}^{p_i-p_1+1}, \tag{3.13}$$

where $a'=(a_1',\ldots,a_N')$.

Proof. By definition of $F_a=F_a(z), F_{a'}=F_{a'}(z)$ one has

$$\sum_{i=1}^{N}a_i|F_a|^{p_i-2}F_a = z = \sum_{i=1}^{N}a_i'|F_{a'}|^{p_i-2}F_{a'}. \tag{3.14}$$

So we have first the estimate

$$\sum_{i=1}^{N}a_i|F_a|^{p_i} = zF_a \leq |z||F_a|$$

and thus

$$\sum_{i=1}^{N}a_i|F_a|^{p_i-1} \leq |z|,$$

which implies

$$|F_a| \leq \left(\frac{|z|}{a_1}\right)^{\frac{1}{p_1-1}}. \tag{3.15}$$

Next by subtraction in (3.14) we derive

$$\sum_{i=1}^{N}a_i\left\{|F_a|^{p_i-2}F_a-|F_{a'}|^{p_i-2}F_{a'}\right\} = \sum_{i=1}^{N}(a_i'-a_i)|F_{a'}|^{p_i-2}F_{a'}.$$

Multiplying both sides by $F_a-F_{a'}$, we get

$$\sum_{i=1}^{N}a_i\left\{|F_a|^{p_i-2}F_a-|F_{a'}|^{p_i-2}F_{a'}\right\}(F_a-F_{a'}) = \sum_{i=1}^{N}(a_i'-a_i)|F_{a'}|^{p_i-2}F_{a'}(F_a-F_{a'}).$$

Recall (see, e.g. [3]) that for $p>1$ there exists a constant $c_p>0$ such that

$$c_p(|\xi|+|\zeta|)^{p-2}|\xi-\zeta|^2 \leq (|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)\cdot(\xi-\zeta), \quad \forall \xi,\zeta\in\mathbb{R}^n. \tag{3.16}$$

Thus, for some constant c_{p_1} we have

$$\begin{aligned}
&c_{p_1} a_1 \{|F_a|+|F_{a'}|\}^{p_1-2}|F_a-F_{a'}|^2 \\
\le &\sum_{i=1}^{N} |a_i'-a_i||F_{a'}|^{p_i-1}|F_a-F_{a'}| \\
\le &\sum_{i=1}^{N} |a_i'-a_i|\{|F_a|+|F_{a'}|\}^{p_i-1}|F_a-F_{a'}|.
\end{aligned}$$

Combining this with (3.15), we get

$$\begin{aligned}
|F_a-F_{a'}| &\le \frac{1}{c_{p_1}a_1}\sum_{i=1}^{N}|a_i'-a_i|\{|F_a|+|F_{a'}|\}^{p_i-p_1+1} \\
&\le \frac{1}{c_{p_1}a_1}\sum_{i=1}^{N}|a_i'-a_i|\left\{\left(\frac{|z|}{a_1}\right)^{\frac{1}{p_1-1}}+\left(\frac{|z|}{a_1'}\right)^{\frac{1}{p_1-1}}\right\}^{p_i-p_1+1}.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Under the assumptions* (3.10), (3.11) *suppose that the $a_i(x,u)$'s are continuous and Lipschitz continuous in u and*

$$0<\lambda\le a_1(x,u),\quad \forall x,u. \tag{3.17}$$

Then (3.12) *admits a unique solution.*

Proof. If u is solution to (3.12) one has

$$\sum_{i=1}^{N} a_i\big(x,u(x)\big)|u'(x)|^{p_i-2}u'(x)=-\int_{\eta_1}^{x} f(s)ds+c, \tag{3.18}$$

where c is some constant. This implies in particular that u' is continuous. Note also that this constant c satisfies

$$|c|\le\int_{\eta_1}^{\eta_2}|f(s)|ds.$$

Indeed, since $u(\eta_1)=u(\eta_2)=0$ there is a point $m\in(\eta_1,\eta_2)$ where $u'(m)=0$ which implies

$$c=\int_{\eta_1}^{m} f(s)ds,$$

and the estimate above follows easily. We claim that this constant c is the same for any solution to (3.12). To show that, let $\tilde{u}$ be solution to (3.12) such that

$$\sum_{i=1}^{N} a_i\big(x,\tilde{u}(x)\big)|\tilde{u}'(x)|^{p_i-2}\tilde{u}'(x) = -\int_{\eta_1}^{x} f(s)ds + c'.$$

Suppose that $c' > c$. Then one has by writing the equations above at η_k, $k=1,2$,

$$\begin{aligned}
&\sum_{i=1}^{N} a_i\big(\eta_k,\tilde{u}(\eta_k)\big)\big|\tilde{u}'(\eta_k)\big|^{p_i-2}\tilde{u}'(\eta_k) \\
&= -\int_{\eta_1}^{\eta_k} f(s)ds + c' > -\int_{\eta_1}^{\eta_k} f(s)ds + c \\
&= \sum_{i=1}^{N} a_i\big(\eta_k,u(\eta_k)\big)\big|u'(\eta_k)\big|^{p_i-2}u'(\eta_k).
\end{aligned}$$

Thus, since $a_i(\eta_k,\tilde{u}(\eta_k)) = a_i(\eta_k,u(\eta_k))$ and the function

$$X \;\to\; \sum_{i=1}^{N} a_i\big(\eta_k,u(\eta_k)\big)|X|^{p_i-2}X$$

is increasing, one gets

$$\tilde{u}'(\eta_k) > u'(\eta_k), \quad k=1,2.$$

This implies that

$$\begin{aligned}
&\tilde{u} > u \quad \text{near} \quad \eta_1, \\
&\tilde{u} < u \quad \text{near} \quad \eta_2,
\end{aligned}$$

recall that $\tilde{u} = u = 0$ at η_1, η_2. Starting from η_1 let us denote by x_0 the first crossing point of the graphs of $\tilde{u}$ and u. At this point one has again

$$\sum_{i=1}^{N} a_i\big(x_0,\tilde{u}(x_0)\big)\big|\tilde{u}'(x_0)\big|^{p_i-2}\tilde{u}'(x_0) > \sum_{i=1}^{N} a_i\big(x_0,u(x_0)\big)\big|u'(x_0)\big|^{p_i-2}u'(x_0),$$

and thus $\tilde{u}'(x_0) > u'(x_0)$. But this would imply, since $\tilde{u}(x_0) = u(x_0)$ that $\tilde{u}(x) < u(x)$ for some $x < x_0$ and a contradiction. If $c' < c$ then swapping $\tilde{u}$ and u would lead to the same contradiction. Thus, if u is solution to (3.12), there exists a fixed constant c such that

$$\sum_{i=1}^{N} a_i\big(x,u(x)\big)|u'(x)|^{p_i-2}u'(x) = -\int_{\eta_1}^{x} f(s)ds + c,$$

i.e. such that

$$u' = F\left(a(x,u(x)), -\int_{\eta_1}^{x} f(s)ds + c\right).$$

We have set $a(x,u(x)) = (a_1(x,u(x)),\ldots,a_N(x,u(x))), F(a,z) = F_a(z)$. It follows from the Lemma 3.1 that $F(a(x,u(x)),-\int_{\eta_1}^{x} f(s)ds+c)$ is Lipschitz continuous in u. Indeed, denoting by K a positive constant bounding $|-\int_{\eta_1}^{x} f(s)ds+c|$, one has by (3.13)

$$\begin{aligned}&\left|F\left(a(x,u(x)),-\int_{\eta_1}^{x} f(s)ds+c\right)-F\left(a(x,v(x)),-\int_{\eta_1}^{x} f(s)ds+c\right)\right|\\ &\le \frac{1}{c_{p_1}\lambda}\sum_{i=1}^{N}\left|a_i(x,u(x))-a_i(x,v(x))\right|\left\{2\left(\frac{K}{a_1}\right)^{\frac{1}{p_1-1}}\right\}^{p_i-p_1+1}.\end{aligned}$$

We have assumed the $a_i(x,u)$'s Lipschitz continuous in u and thus

$$F\left(a(x,u(x)),-\int_{\eta_1}^{x} f(s)ds+c\right)$$

is also Lipschitz continuous in u. Since u solution to (3.12) satisfies

$$\begin{cases} u' = F\left(a(x,u(x)),-\int_{\eta_1}^{x} f(s)ds+c\right), & x\in(\eta_1,\eta_2),\\ u(\eta_1)=0, \end{cases}$$

u is unique. This completes the proof of the theorem. □

Remark 3.1. In the case where $f>0$ then by (3.12),

$$x \;\to\; \sum_{i=1}^{N} a_i(u(x))|u'(x)|^{p_i-2}u'(x)$$

is decreasing and thus vanishes at exactly one point where the maximum of u is.

We turn now to the results that we are able to prove in higher dimensions. We consider first a peculiar example.

Theorem 3.3. *Suppose that there exist functions $\alpha_i=\alpha_i(x)$ and a continuous function $b(u)$ such that for positive constants $\lambda_0,\lambda_1,b_0,b_1$ one has*

$$\begin{aligned} &\lambda_0\le\alpha_i(x)\le\lambda_1 \quad a.e. \quad x\in\Omega,\\ &b_0\le b(u)\le b_1, \quad \forall u\in\mathbb{R},\\ &a_i(x,u)=\alpha_i(x)b(u)^{p_i-1} \end{aligned}\tag{3.19}$$

for all $i=1,\dots,N$, *then* (1.3) *admits at most one solution. More generally if* $u_k, k=1,2$ *denotes a solution to* (1.3) *corresponding to* $f=f_k$ *then*

$$f_1 \le f_2 \quad \textit{implies} \quad u_1 \le u_2.$$

$f_1 \le f_2$ *means, as usual in this context,* $\langle f_1 - f_2, v\rangle \le 0$ *for any* $v \in W_0^{1,p_N}(\Omega), v \ge 0$.

Proof. If u_k is solution to (1.3) corresponding to $f=f_k$, one sets

$$U_k(x) = \int_0^{u_k(x)} b(s)\,ds.$$

Then clearly

$$\begin{aligned}&\nabla U_k(x) = b(u_k)\nabla u_k(x),\\ &|\nabla U_k(x)|^{p_i-2}\nabla U_k(x) = b(u_k)^{p_i-1}|\nabla u_k(x)|^{p_i-2}\nabla u_k(x),\end{aligned}$$

in such a way that U_k satisfies for $k=1,2$,

$$\begin{cases} U_k \in W_0^{1,p_N}(\Omega), \\ \displaystyle\int_\Omega \sum_{i=1}^N \alpha_i(x)|\nabla U_k|^{p_i-2}\nabla U_k \cdot \nabla v\,dx = \langle f_k, v\rangle, \quad \forall v \in W_0^{1,p_N}(\Omega). \end{cases}$$

By subtraction one gets

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \alpha_i(x)\Big(|\nabla U_1|^{p_i-2}\nabla U_1 - |\nabla U_2|^{p_i-2}\nabla U_2\Big)\cdot\nabla v\,dx\\ &= \langle f_1 - f_2, v\rangle, \quad \forall v \in W_0^{1,p_N}(\Omega).\end{aligned}$$

Taking $v=(U_1-U_2)^+$ the positive part of U_1-U_2 one gets easily for some constants $c_i>0$ (see (3.16))

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \alpha_i(x)c_i(|\nabla U_1|+|\nabla U_2|)^{p_i-2}|\nabla(U_1-U_2)^+|^2dx\\ &\le \int_\Omega \sum_{i=1}^N \alpha_i(x)\Big(|\nabla U_1|^{p_i-2}\nabla U_1 - |\nabla U_2|^{p_i-2}\nabla U_2\Big)\cdot\nabla(U_1-U_2)^+dx \le 0.\end{aligned}$$

This implies that $(U_1-U_2)^+=0$ and thus $U_1 \le U_2$ which is equivalent to $u_1 \le u_2$. Uniqueness follows by choosing $f=f_1=f_2$. This completes the proof of the theorem. □

Remark 3.2. Note that only one of the α_i's needs here to be positive, for instance α_N if one wants to rely on (1.1) to have existence of a solution.

In the case of one single operator, i.e. $N=1$ the above theorem can be rephrased as follows.

Theorem 3.4. *Let α_1 be such $0<\lambda_0\leq\alpha_1(x)\leq\lambda_1$ and $a(u)$ be a continuous function such that for some positive constants*

$$b_0\leq a(u)\leq b_1. \tag{3.20}$$

For $p>1$ consider u solution to

$$\begin{cases} u\in W_0^{1,p}(\Omega), \\ \int_\Omega \alpha_1(x)a(u)|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx=\langle f,v\rangle, \quad \forall v\in W_0^{1,p}(\Omega). \end{cases} \tag{3.21}$$

Then (3.21) *admits a unique solution. Moreover, if $u_k,k=1,2$ denotes a solution to* (3.21) *corresponding to f_k then*

$$f_1\leq f_2 \quad \textit{implies} \quad u_1\leq u_2.$$

If $a_k,k=1,2$ denotes a function a satisfying (3.20) *and if u_k, $k=1,2$ denotes a solution to* (3.21) *corresponding to a_k,f_k then*

$$0\leq f_1\leq f_2, \quad a_1\geq a_2 \quad \textit{implies} \quad u_1\leq u_2.$$

Proof. The first part of the theorem follows from Theorem 3.3 (see also the Remark 3.2) by setting $b(u)=a(u)^{1/(p-1)}$.

For the second part of the theorem note first that, since the f_k are nonnegative, one has $u_k\geq 0$ for $k=1,2$. This is a consequence of the first part of the theorem. Set as previously

$$U_k(x)=\int_0^{u_k(x)} a_k(s)^{\frac{1}{p-1}}\,ds.$$

As in the proof of Theorem 3.3 one notices that U_k satisfies for $k=1,2$,

$$\begin{cases} U_k\in W_0^{1,p}(\Omega), \\ \int_\Omega \alpha_1(x)|\nabla U_k|^{p-2}\nabla U_k\cdot\nabla v\,dx=\langle f_k,v\rangle, \quad \forall v\in W_0^{1,p}(\Omega). \end{cases}$$

By subtraction we get

$$\int_\Omega \alpha_1(x)\big(|\nabla U_1|^{p-2}\nabla U_1-|\nabla U_2|^{p-2}\nabla U_2\big)\cdot\nabla v\,dx=\langle f_1-f_2,v\rangle, \quad \forall v\in W_0^{1,p}(\Omega).$$

Taking $v=(U_1-U_2)^+$ one deduces as above that $U_1\le U_2$, i.e.

$$\begin{aligned} U_1(x)&=\int_0^{u_1(x)} a_1(s)^{\frac{1}{p-1}}\,ds\le U_2(x)\\ &=\int_0^{u_2(x)} a_2(s)^{\frac{1}{p-1}}ds\le\int_0^{u_2(x)} a_1(s)^{\frac{1}{p-1}}ds, \end{aligned}\tag{3.22}$$

since $a_1\ge a_2$ and $u_2\ge 0$. The result follows since

$$\int_0^{u_1(x)} a_1(s)^{\frac{1}{p-1}}ds\le\int_0^{u_2(x)} a_1(s)^{\frac{1}{p-1}}ds$$

is equivalent to $u_1\le u_2$. This completes the proof of the theorem. □

Remark 3.3. Note that without the positivity of f_k one gets nevertheless a comparison principle, i.e. $U_1\le U_2$ (cf. (3.22)) and only the positivity of f_2 is used subsequently. It is interesting to see that the monotonicity result is here at two levels f and a and that one does not need any Lipschitz continuity on a.

4 Concluding remarks

The same results as above hold for instance for the problems of the type

$$\begin{cases} u\in W_0^{1,p_N}(\Omega),\\ -\sum_{i=1}^N \partial_{x_i}\Big(a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\Big)=f \quad \text{in } \Omega. \end{cases}$$

In fact, the two operators, i.e. the one just above and the one in (1.2), coincide in dimension one.

In higher dimensions we suspect that the result obtained in the case where $N=1$ go through for any N when

$$a_i(x,u)=\alpha_i(x)a_i(u),$$

$a_i(u)$ being continuous, bounded and bounded away from 0. However, so far, we have been unable to show it except in the particular case of Theorem 3.3.

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