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Generalized Isospectral-Nonisospectral Modified Korteweg-de Vries Integrable Hierarchies and Related Properties

Huanhuan Lu, Xinan Ren*, Yufeng Zhang and Hongyi Zhang

School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, China

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Abstract. In this article, a new technique for deriving integrable hierarchy is discussed, i.e., such that are derived by combining the Tu scheme with the vector product. Several classes of spectral problems are introduced by three-dimensional loop algebra and six-dimensional loop algebra whose commutators are vector product, and the six-dimensional loop algebra is derived from the enlargement of the three-dimensional loop algebra. It is important that we make use of the variational method to create a new vector-product trace identity for which the Hamiltonian structure of the isospectral integrable hierarchy is worked out. The derived integrable hierarchies are reduced to the modified Korteweg-de Vries (mKdV) equation, generalized coupled mKdV integrable system and non-isospectral mKdV equation under specific parameter selection. Starting from a 3×3 matrix spectral problem, we subsequently construct an explicit N-fold Darboux transformation for integrable system (2.8) with the help of a gauge transformation of the corresponding spectral problem. At the same time, the determining equations of nonclassical symmetries associated with mKdV equation are presented in this paper. It follows that we investigate the coverings and the nonlocal symmetries of the nonisospectral mKdV equation by applying the classical Frobenius theorem and the coordinates of a infinitely-dimensional manifold in the form of Cartesian product.

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*Corresponding author.
Email: renx@cumt.edu.cn (X. Ren), lhh1263910575@163.com (H. Lu)

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1 Introduction

It has been a hot topic in searching for new integrable systems. Since Magri [1] proposed the Lax pair method for generating integrable equations, a lot of experts and scholars have devoted them to this research field so that a variety of modified methods have been developed in the past few years [2–8]. One of the most classical method [6] called Tu scheme by Ma [7] applies Lax pairs adjoint with finite-dimensional Lie algebras to investigate integrable equations. Before we present the scheme, a few of basic notations are first introduced.

Let G be a matrix Lie algebra over the complex field C and $\tilde{G}=G\otimes C(\lambda,\lambda^{-1})$ be its loop algebra, where $C(\lambda,\lambda^{-1})$ is the set of Laurent polynomials in λ. The gradation of $\tilde{G}$ is taken by $deg(x\otimes\lambda^n)=n, x\in G$. Let $g\in\tilde{G}$ and $g=\sum_n g_n$, $deg g_n=n$, be its gradation decomposition. Set $g_+=\sum_{n\geq 0}g_n$, we consider the isospectral problem

$$\varphi_x=U(u,\lambda)\varphi,$$

with $U=U(u,\lambda)=e_0(\lambda)+u_1e_1(\lambda)+\cdots+u_pe_p(\lambda)$, where $u=(u_1,\cdots,u_p)$ is a potential function, $e_0(\lambda),e_1(\lambda),\cdots,e_p(\lambda)\in\tilde{G}$. Suppose $e_0,e_1,\cdots,e_p$ are linearly independent and $\varepsilon_0>0$, $\varepsilon_0>\varepsilon_i$, $i=1,\cdots,p$; here $\varepsilon_i=deg e_i$. The explicit steps of Tu scheme for generating Lax integrable systems are as follows:

First, we take a solution $V=V(\lambda)$ of the equation

$$V_x(\lambda)=[U(\lambda),V(\lambda)].$$

Second, we search for $\Delta_n\in\tilde{G}$ so that, for $V^{(n)}=(\lambda^nV)_++\Delta_n$, it holds that

$$V_x^{(n)}-[U,V^{(n)}]=Ce_1+\cdots+Ce_p.$$

This requirement yields a hierarchy of evolution equations

$$U_{t_n}=V_x^{(n)}-[U,V^{(n)}]. \tag{1.1}$$

Finally, using the trace identity

$$\frac{\delta}{\delta u_i}\left\langle V,\frac{\partial U}{\partial\lambda}\right\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V,\frac{\partial U}{\partial u_i}\right\rangle$$

deduces the generalized Hamiltonian structure of (1.1), where $\langle x,y\rangle=tr(xy)$, $x,y\in\tilde{G}$.

By utilizing the scheme, many interesting integrable hierarchies of evolution equations and the corresponding Hamiltonian structures were obtained [9–13]. Also applying Lax pair method, Li Yishen [14, 15] proposed a way for generating non-isospectral integrable hierarchies, which means the spectral parameter λ is given by

$$\lambda_t=\sum_{m=0}^{n}k_m(t)\lambda^m.$$

It follows that Ma and Qiao presented some nonisospectral integrable hierarchies in the special case $\lambda_t=M(t)\lambda^n$ [4, 5]. Recently, many nonisospectral super or multi-component integrable hierarchies were followed to produce in [16–20]. It follows that some properties of integrable hierarchies and reduced equation can be discussed in terms of the approaches introduced in [21–26] (such as Bäcklund transformation, Darboux transformation, symmetries and so on). More recently, Zhang et al. [27] proposed a new technique to derive isospectral and nonisospectral integrable hierarchies. Different from previous methods, the method mainly uses vector-product loop algebra to introduce linear spectral problems. At the same time, the normal zero curvature equation is replaced by the vector product zero curvature equation.

Here, it is quite necessary to recall the basic notations and computing formulae.

Definition 1.1. *The outer product (also called vector product) of two vectors $\vec{a}$ and $\vec{b}$ in the space R^3 is denoted by $\vec{a}\times\vec{b}$ whose length is given by*

$$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\langle(\vec{a},\vec{b})\rangle,$$

whose direction is required to be vertical to the vector $\vec{a}$ and $\vec{b}$, further satisfies the right-handed coordinate system. For arbitrary vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, the vector product possesses the following properties

(1) $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$,

(2) $(\lambda\vec{a})\times\vec{b}=\lambda(\vec{a}\times\vec{b})$,

(3) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$, $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.

Assume $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are unit coordinate vectors in the right-handed vectical coordinate system $\{0;\vec{e}_1,\vec{e}_2,\vec{e}_3\}$, then for arbitrary vectors $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)\in\{0;\vec{e}_1,\vec{e}_2,\vec{e}_3\}$, where a_i, b_i are coordinates of the vectors $\vec{a}$ and $\vec{b}$, respectively, the vector product of the vectors $\vec{a}$, $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=det\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}.$$

Denote a vector space by $\mathcal{S}$, that is

$$\mathcal{S}=span\{\vec{e}_1,\vec{e}_2,\vec{e}_3\},$$

then arbitrary a row vector space $\mathcal{W}$ with three dimensional is isomorphic to the space $\mathcal{S}$. Therefore, for arbitrary vectors $\vec{a},\vec{b}\in\mathcal{W}$, we define a commutation operation by vector product

$$[\vec{a},\vec{b}]_\nu=\vec{a}\times\vec{b}. \tag{1.2}$$

It can be verified that $\mathcal{W}$ is a Lie algebra. That is,

Theorem 1.1. *For three dimensional row-vector space $\mathcal{W}$, if $\vec{a},\vec{b}\in\mathcal{W}$ satisfy* (1.2), *then $\mathcal{W}$ is a Lie algebra.*

Theorem 1.2. *For arbitrary vectors $\vec{U},\vec{V},\vec{\phi}\in\mathcal{W}$, suppose that*

$$\vec{\phi}_x=\vec{U}\times\vec{\phi},\quad \vec{\phi}_t=\vec{V}\times\vec{\phi}, \tag{1.3}$$

then the compatibility condition of (1.3) *leads to a new form of zero curvature equation*

$$\vec{U}_t-\vec{V}_x+\vec{U}\times\vec{V}=\vec{0}, \tag{1.4}$$

which is also called a vector-product zero curvature equation.

The detailed proof of Theorems 1.1 and 1.2 has been shown in [27].

The aim of this paper is to generate new integrable hierarchies with the aid of vector product zero curvature equation and vector product loop algebra. To be specific, by means of the Lie algebra S, we construct the isospectral and nonisospectral modified Korteweg-de Vries (mKdV) integrable hierarchies. Among them, the derived isospectral integrable hierarchy is reduced to the generalized mKdV system (2.8). Under specific parameter selection, the generalized mKdV integrable system is further reduced to the mKdV equation (2.9) and generalized coupled mKdV integrable system (2.10). In particularly, we discuss in detail the Darboux transformation for (2.8) and nonclassical symmetries for (2.9). Meanwhile, the nonisospectral mKdV integrable hierarchy is reduced to the nonisospectral mKdV equation whose nonlocal symmetry are also presented. In the last section, the high-dimensional vector-product Lie algebra G equipped with two loop algebras are applied to derived some integrable hierarchies which can again be reduced to the mKdV equation and nonisospectral mKdV equation which differ only in coefficients from those obtained in Section 2.

2 Isospectral and nonisospectral modified Korteweg-de Vries integrable hierarchies

2.1 Isospectral integrable hierarchy

Firstly, we show a Lie algebra $\mathcal{S}=span\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$, where $\vec{e}_j(n)=\vec{e}_j\lambda^n$, and along with the commutators

$$\vec{e}_1(m)\times\vec{e}_2(n)=\vec{e}_3(m+n),\quad \vec{e}_2(m)\times\vec{e}_3(n)=\vec{e}_1(m+n),\quad \vec{e}_3(m)\times\vec{e}_1(n)=\vec{e}_2(m+n).$$

Consider the following spectral problems

$$\vec{\phi}_x=\vec{U}\times\vec{\phi},\qquad \vec{U}=2\vec{e}_1(1)+q\vec{e}_2(0)+r\vec{e}_3(-1)+s\vec{e}_1(-1), \tag{2.1a}$$

$$\vec{\phi}_t=\vec{V}\times\vec{\phi},\qquad \vec{V}=\sum_{m\geq 0}(a_m\vec{e}_1(-m)+b_m\vec{e}_2(-m)+c_m\vec{e}_3(-m)). \tag{2.1b}$$

Solving the station zero curvature equation

$$\vec{V}_x=\vec{U}\times\vec{V} \tag{2.2}$$

yields

$$a_{m,x}=qc_m-rb_{m-1}, \tag{2.3a}$$

$$b_{m,x}=-2c_{m+1}+ra_{m-1}-sc_{m-1}, \tag{2.3b}$$

$$c_{m,x}=2b_{m+1}-qa_m+sb_{m-1}. \tag{2.3c}$$

Take $b_0=c_0=0$, $a_0=\alpha(t)$, $a_1=c_1=0$, then one has

$$b_1=\frac{\alpha q}{2},\quad b_2=0,\quad c_2=\frac{\alpha r}{2}-\frac{\alpha q_x}{4},\quad a_2=-\frac{\alpha q^2}{8}, \tag{2.4a}$$

$$b_3=\frac{\alpha}{4}r_x-\frac{\alpha}{8}q_{xx}-\frac{\alpha}{16}q^3-\frac{\alpha}{4}qs,\quad c_3=0, \tag{2.4b}$$

$$c_4=-\frac{\alpha}{16}q^2r-\frac{\alpha}{4}rs+\frac{\alpha}{8}q_xs-\frac{\alpha}{8}r_{xx}+\frac{\alpha}{16}q_{xxx}+\frac{3\alpha}{32}q^2q_x+\frac{\alpha}{8}(qs)_x,\cdots, \tag{2.4c}$$

$$a_{2k+1}=c_{2k+1}=0,\quad b_{2k+2}=0. \tag{2.4d}$$

Note that

$$\begin{aligned}\vec{V}_+^{(n)}&=\sum_{m=0}^{n}(a_m\vec{e}_1(-m)+b_m\vec{e}_2(-m)+c_m\vec{e}_3(-m))\lambda^n=\lambda^n\vec{V}-\vec{V}_-^{(n)}\\&=\lambda^n\vec{V}-\sum_{m=n+1}^{\infty}(a_m\vec{e}_1(-m)+b_m\vec{e}_2(-m)+c_m\vec{e}_3(-m))\lambda^n,\end{aligned}$$

then (2.2) can be broken down into

$$-\vec{V}^{(n)}_{+,x}+\vec{U}\times\vec{V}^{(n)}_{+}=(\vec{V}^{(n)}_{-})_x-\vec{U}\times\vec{V}^{(n)}_{-}. \tag{2.5}$$

We find that the gradation of the left-hand side of (2.5) is more than -1, while the right-hand side is less than 0. Therefore, taking the gradations 0 and -1 in (2.5), one has

$$\begin{aligned}-\vec{V}^{(n)}_{+,x}+\vec{U}\times\vec{V}^{(n)}_{+}=&-rb_n\vec{e}_1(-1)+(ra_n-sc_n)\vec{e}_2(-1)+sb_n\vec{e}_3(-1)\\&-2b_{n+1}\vec{e}_3(0)+2c_{n+1}\vec{e}_2(0).\end{aligned}$$

Hence, choose $n=2k+1$, the isospectral zero curvature equation

$$\frac{\partial\vec{U}}{\partial u}u_t-\vec{V}^{(n)}_{+,x}+\vec{U}\times\vec{V}^{(n)}_{+}=\vec{0} \tag{2.6}$$

leads to the isospectral mKdV integrable hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ s\end{pmatrix}_t=\begin{pmatrix} -2c_{2k+2}\\ -sb_{2k+1}\\ rb_{2k+1}\end{pmatrix}. \tag{2.7}$$

When $k=1$, the hierarchy (2.7) reduces to

$$q_t=\frac{\alpha}{8}q^2r+\frac{\alpha}{2}rs-\frac{\alpha}{4}q_xs+\frac{\alpha}{4}r_{xx}-\frac{\alpha}{8}q_{xxx}-\frac{3\alpha}{16}q^2q_x-\frac{\alpha}{4}(qs)_x, \tag{2.8a}$$

$$r_t=-\frac{\alpha}{4}r_xs+\frac{\alpha}{8}q_{xx}s+\frac{\alpha}{16}q^3s+\frac{\alpha}{4}qs^2, \tag{2.8b}$$

$$s_t=\frac{\alpha}{4}rr_x-\frac{\alpha}{8}q_{xx}r-\frac{\alpha}{16}q^3r-\frac{\alpha}{4}qsr. \tag{2.8c}$$

Specially, set $s=0$, $r=0$, (2.8) reduces to

$$q_t=-\frac{\alpha}{8}q_{xxx}-\frac{3\alpha}{16}q^2q_x, \tag{2.9}$$

which is modified mKdV equation.

Setting $s=ir$, (2.8) reduces to generalized coupled mKdV integrable system

$$q_t=\alpha\left(\frac{1}{8}q^2r+\frac{i}{2}r^2-\frac{i}{4}q_xr+\frac{1}{4}r_{xx}-\frac{1}{8}q_{xxx}-\frac{3}{16}q^2q_x-\frac{i}{4}(qr)_x\right), \tag{2.10a}$$

$$r_t=\alpha\left(-\frac{i}{4}rr_x+\frac{i}{8}q_{xx}r+\frac{i}{16}q^3r-\frac{1}{4}qr^2\right). \tag{2.10b}$$

2.1.1 Hamiltonian structure for isospectral integrable hierarchy (2.7)

The existing literature shows that the Hamiltonian can be furnished for the resulting integrable hierarchies by applying the trace identity [6] and the quadratic-type identity [28,29]. In the section, we follow the idea for deriving the quadratic-form identity to deduce a new vector-product identity for generating Hamiltonian structure of the isospectral integrable hierarchies (2.10) obtained by vector-product equation (1.4).

Assume $\vec{a},\vec{b},\vec{c}\in\mathcal{W}$, a linear functional $\{\vec{a},\vec{b}\}_v=:\vec{a}F\vec{b}^T$, where F is a square matrix with constant entries and the functional $\{\vec{a},\vec{b}\}_v$ satisfies the following relations

$$\{\vec{a},\vec{b}\}_\nu=\{\vec{b},\vec{a}\}_\nu, \tag{2.11a}$$

$$\{\vec{a},[\vec{b},\vec{c}]_\nu\}_\nu=\{[\vec{a},\vec{b}]_\nu,\vec{c}\}_\nu, \tag{2.11b}$$

Eq. (2.11a) requires that F is symmetry, i.e., $F^T=F$. Since the vector product of $\vec{a}$ and $\vec{b}$ can be written as

$$\begin{aligned}\{\vec{a},\vec{b}\}_\nu=\vec{a}\times\vec{b}&=(a_2b_3-a_3b_2,a_3b_1-b_3a_1,a_1b_2-b_1a_2)\\&=(a_1,a_2,a_3)\begin{pmatrix}0&-b_3&b_2\\b_3&0&b_1\\-b_2&b_1&0\end{pmatrix}=:\vec{a}R(b),\end{aligned} \tag{2.12}$$

where $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$. Therefore, (2.11b) requires that

$$R(b)F=-(R(b)F)^T. \tag{2.13}$$

In terms of (2.13), a direct calculation produces

$$F=\begin{pmatrix}\rho&\rho&\rho\\\rho&\rho&\rho\\\rho&\rho&\rho\end{pmatrix},$$

where ρ is a constant. Noting

$$\vec{V}=(a,b,c),\quad \vec{U}_\lambda=\left(2-\frac{s}{\lambda^2},0,-\frac{1}{\lambda^2}\right),$$

then we have

$$\left\{\vec{V},\frac{\partial\vec{U}}{\partial\lambda}\right\}_\nu=\vec{V}F\left(\frac{\partial\vec{U}}{\partial\lambda}\right)^T=2a\rho-\frac{as\rho}{\lambda^2}-\frac{cr\rho}{\lambda^2},\quad \left\{\vec{V},\frac{\partial\vec{U}}{\partial q}\right\}_\nu=\vec{V}F\left(\frac{\partial\vec{U}}{\partial q}\right)^T=b\rho,$$

$$\left\{\vec{V},\frac{\partial\vec{U}}{\partial r}\right\}_\nu=\vec{V}F\left(\frac{\partial\vec{U}}{\partial r}\right)^T=\frac{c\rho}{\lambda},\qquad \left\{\vec{V},\frac{\partial\vec{U}}{\partial s}\right\}_\nu=\vec{V}F\left(\frac{\partial\vec{U}}{\partial s}\right)^T=\frac{a\rho}{\lambda}.$$

Substituting the above computations into the following vector-product trace identity

$$\frac{\delta}{\delta u}\left\{\vec{V},\vec{U}_\lambda\right\}_\nu=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^\gamma\left\{\vec{V},\vec{U}_u\right\}_\nu$$

gives

$$\frac{\delta}{\delta u}\left(2a\rho-\frac{as\rho}{\lambda^2}-\frac{cr\rho}{\lambda^2}\right)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^\gamma\left(\begin{array}{c} b\rho \\ \dfrac{c\rho}{\lambda} \\ \dfrac{a\rho}{\lambda}\end{array}\right),$$

where

$$a=\sum_{m\geq 0}a_m\lambda^{-m},\quad b=\sum_{m\geq 0}b_m\lambda^{-m},\quad c=\sum_{m\geq 0}c_m\lambda^{-m}.$$

Comparing the coefficients of λ^{-n-2}, we get

$$\frac{\delta}{\delta u}(2a_{n+2}-sa_n-rc_n)=(\gamma-n-1)\left(\begin{array}{c} b_{n+1} \\ c_n \\ a_n\end{array}\right). \tag{2.14}$$

By substituting initial values of (2.4), one can find $\gamma=0$ and thus obtain the Hamiltonian structure

$$\frac{\delta H_n}{\delta u}=\left(\begin{array}{c} b_{n+1} \\ c_n \\ a_n\end{array}\right),\quad H_n=\frac{2a_{n+2}-sa_n-rc_n}{-n-1}.$$

It follows that (2.7) can be written as

$$u_t=\left(\begin{array}{c} q \\ r \\ s\end{array}\right)_t=J\left(\begin{array}{c} b_{2k+1} \\ c_{2k} \\ a_{2k}\end{array}\right)=J\frac{\delta H_{2k}}{\delta u}, \tag{2.15}$$

where

$$J=\left(\begin{array}{ccc} \partial & s & -r \\ -s & 0 & 0 \\ r & 0 & 0\end{array}\right),$$

and J is a Hamiltonian operator.

2.1.2 Darboux transformation for (2.8)

In this section, we shall follow Neugebauer's scheme [30–32] to construct the N-fold Darboux transformation for (2.8). Before that, the Lax pair for (2.8) should be worked out. Based on the purpose, we consider the classical Lie algebra $W=\{h,e,f\}$ that is isomorphic to S, where

$$h=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\quad e=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad f=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$h(n)=h\lambda^n,\quad e(n)=e\lambda^n,\quad f(n)=f\lambda^n.$$

By means of the Lie algebra W, the 3×3 spectral problem corresponding to integrable system (2.8) should be expressed as

$$\varphi_x=U(q,r,s,\lambda)\varphi=\begin{pmatrix} 0 & -r\lambda^{-1} & q \\ r\lambda^{-1} & 0 & -(2\lambda+s\lambda^{-1}) \\ -q & 2\lambda+s\lambda^{-1} & 0 \end{pmatrix}\varphi, \tag{2.16a}$$

$$\varphi_t=V(q,r,s,\lambda)\varphi=\begin{pmatrix} 0 & -\left(\frac{\alpha}{2}r-\frac{\alpha}{4}q_x\right)\lambda & \frac{\alpha}{2}q\lambda^2+\frac{\alpha}{4}r_x-\frac{\alpha}{8}q_{xx}-\frac{\alpha}{16}q^3-\frac{\alpha}{4}qs \\ \left(\frac{\alpha}{2}r-\frac{\alpha}{4}q_x\right)\lambda & 0 & -\alpha(t)\lambda^3+\frac{\alpha}{8}q^2\lambda \\ -\frac{\alpha}{2}q\lambda^2-\left(\frac{\alpha}{4}r_x-\frac{\alpha}{8}q_{xx}-\frac{\alpha}{16}q^3-\frac{\alpha}{4}qs\right) & \alpha(t)\lambda^3-\frac{\alpha}{8}q^2\lambda & 0 \end{pmatrix}\varphi, \tag{2.16b}$$

whose compatibility condition gives rise to (2.8). Consider a matrix T that satisfies the gauge transformation

$$\bar{\varphi}=T\varphi. \tag{2.17}$$

A new spectral problem reads

$$\bar{\varphi}_x=\bar{U}\bar{\varphi},\qquad \bar{U}=(T_x+TU)T^{-1}, \tag{2.18a}$$

$$\bar{\varphi}_t=\bar{V}\bar{\varphi},\qquad \bar{V}=(T_t+TV)T^{-1}, \tag{2.18b}$$

where $\bar{U}$ and $\bar{V}$ have the same forms as U and V, respectively. Meanwhile, the old potential q and r in U and V will be mapped into new potential $\bar{q}$ and $\bar{r}$ in $\bar{U}$ and $\bar{V}$. The process can be iterated and usually it yields a series of multi-soliton solutions.

Let

$$\begin{aligned}
\varphi(\lambda)&=(\varphi_1(\lambda),\varphi_2(\lambda),\varphi_3(\lambda))^T,\\
\psi(\lambda)&=(\psi_1(\lambda),\psi_2(\lambda),\psi_3(\lambda))^T,\\
\chi(\lambda)&=(\chi_1(\lambda),\chi_2(\lambda),\chi_3(\lambda))^T,
\end{aligned}$$

be three basic solutions of (2.16). Based on that, we define the following algebraic system

$$\sum_{i=0}^{n-1}\underset{i}{a_{11}}\lambda_j^i+\alpha_j^{(1)}\sum_{i=0}^{n-1}\underset{i}{a_{12}}\lambda_j^i+\alpha_j^{(2)}\sum_{i=0}^{n-1}\underset{i}{a_{13}}\lambda_j^i=-\lambda_j^n, \tag{2.19a}$$

$$\begin{aligned}
&\sum_{i=0}^{n-2}\underset{i}{a_{21}}\lambda_j^i+\alpha_j^{(1)}\sum_{i=0}^{n-1}\underset{i}{a_{22}}\lambda_j^i+\alpha_j^{(2)}\sum_{i=0}^{n-2}\underset{i}{a_{23}}\lambda_j^i\\
&=\underset{n-1}{a_{12}}\lambda_j^{n-1}-\alpha_j^{(1)}\lambda_j^n-\alpha_j^{(2)}\underset{n}{a_{23}}\lambda_j^n, \qquad 1\le j\le 3n,
\end{aligned}\tag{2.19b}$$

$$\begin{aligned}
&\sum_{i=0}^{n-3}\underset{i}{a_{31}}\lambda_j^i+\alpha_j^{(1)}\sum_{i=0}^{n-2}\underset{i}{a_{32}}\lambda_j^i+\alpha_j^{(2)}\sum_{i=0}^{n-2}\underset{i}{a_{33}}\lambda_j^i\\
&=-\underset{n-1}{a_{13}}\lambda_j^{n-1}+\underset{n-2}{a_{13}}\lambda_j^{n-2}+\alpha_j^{(1)}\underset{n}{a_{23}}\lambda_j^n-\alpha_j^{(2)}\lambda_j^n-\alpha_j^{(2)}\underset{n-1}{a_{22}}\lambda_j^{n-1}, \qquad 1\le j\le 3n,
\end{aligned}\tag{2.19c}$$

with

$$\alpha_j^{(1)}=\frac{\varphi_2+\gamma_j^{(1)}\psi_2+\gamma_j^{(2)}\chi_2}{\varphi_1+\gamma_j^{(1)}\psi_1+\gamma_j^{(2)}\chi_1}, \quad \alpha_j^{(2)}=\frac{\varphi_3+\gamma_j^{(1)}\psi_3+\gamma_j^{(2)}\chi_3}{\varphi_1+\gamma_j^{(1)}\psi_1+\gamma_j^{(2)}\chi_1}, \quad 1\le j\le 3n, \tag{2.20}$$

where $\gamma_j^{(1)}$ and $\gamma_j^{(2)}$ are some suitable parameters, which ensure the determinants of the coefficients for (2.19) are nonzero.

Thus, the transformation matrix T has the following form

$$T=\begin{pmatrix}
\lambda^n+\sum\limits_{i=0}^{n-1}\underset{i}{a_{11}}\lambda_j^i & \sum\limits_{i=0}^{n-1}\underset{i}{a_{12}}\lambda^i & \sum\limits_{i=0}^{n-1}\underset{i}{a_{13}}\lambda_j^i\\
-\underset{n-1}{a_{12}}\lambda^{n-1}+\sum\limits_{i=0}^{n-2}\underset{i}{a_{21}}\lambda^i & \lambda^n+\sum\limits_{i=0}^{n-1}\underset{i}{a_{22}}\lambda_j^i & \underset{n}{a_{23}}\lambda^n+\sum\limits_{i=0}^{n-2}\underset{i}{a_{23}}\lambda^i\\
\underset{n-1}{a_{13}}\lambda^{n-1}-\underset{n-2}{a_{13}}\lambda^{n-2}+\sum\limits_{i=0}^{n-3}\underset{i}{a_{31}}\lambda^i & -\underset{n}{a_{23}}\lambda^n+\sum\limits_{i=0}^{n-2}\underset{i}{a_{32}}\lambda^i & \lambda^n+\underset{n-1}{a_{22}}\lambda^{n-1}+\sum\limits_{i=0}^{n-2}\underset{i}{a_{33}}\lambda_j^i
\end{pmatrix}, \tag{2.21}$$

which can be rewritten as

$$T=T_n\lambda^n+T_{n-1}\lambda^{n-1}+T_{n-2}\lambda^{n-2}+\sum_{i=0}^{n-2}T_i\lambda^i, \tag{2.22}$$

where

$$T_n=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \underset{n}{a_{23}} \\ 0 & -\underset{n}{a_{23}} & 1 \end{pmatrix}, \qquad T_{n-1}=\begin{pmatrix} \underset{n-1}{a_{11}} & \underset{n-1}{a_{12}} & \frac{1}{2}\underset{n}{a_{23}}\underset{n-1}{a_{12}} \\ -\underset{n-1}{a_{12}} & \underset{n-1}{a_{22}} & 0 \\ \frac{1}{2}\underset{n}{a_{23}}\underset{n-1}{a_{12}} & 0 & \underset{n-1}{a_{22}} \end{pmatrix}, \tag{2.23a}$$

$$T_{n-2}=\begin{pmatrix} \underset{n-2}{a_{11}} & \underset{n-2}{a_{12}} & \underset{n-2}{a_{13}} \\ \underset{n-2}{a_{21}} & \underset{n-2}{a_{22}} & \underset{n-2}{a_{23}} \\ -\underset{n-1}{a_{13}} & \underset{n-2}{a_{32}} & \underset{n-2}{a_{33}} \end{pmatrix}, \quad T_i=\begin{pmatrix} \underset{i}{a_{11}} & \underset{i}{a_{12}} & \underset{i}{a_{13}} \\ \underset{i}{a_{21}} & \underset{i}{a_{22}} & \underset{i}{a_{23}} \\ \underset{i}{a_{31}} & \underset{i}{a_{32}} & \underset{i}{a_{33}} \end{pmatrix}, \quad i=0,1,\cdots,n-3. \tag{2.23b}$$

Eqs. (2.19) and (2.22) implies that $detT$ is the $3n$-th-order polynomial in λ, and λ_j $(1\leq j\leq 3n)$ are all its roots. As a consequence, we have

$$detT(\lambda)=\Pi_{j=1}^{3n}(\lambda-\lambda_j). \tag{2.24}$$

Substituting (2.22) into the first equation in (2.18) and comparing the coefficients of λ^{n+1}, λ^n, λ^{n-1} and λ^{n-2} give rises to

$$\bar{q}=q-2\underset{n-1}{a_{12}}, \quad \bar{r}=r-2\underset{n-2}{a_{13}}-\underset{n-1}{a_{12,x}}, \quad \bar{s}=s+\underset{n-1}{a_{22,x}}-2\underset{n-2}{a_{23}}-2\underset{n-2}{a_{32}}, \tag{2.25}$$

and

$$-\bar{r}\underset{n}{a_{23}}+\bar{q}\underset{n-1}{a_{33}}=\frac{1}{2}(\underset{n}{a_{23}}\underset{n-1}{a_{12}})_x+q\underset{n-1}{a_{11}}-2\underset{n-2}{a_{12}}, \quad -2\underset{n-2}{a_{32}}+\bar{s}\underset{n}{a_{32}}=\underset{n-1}{a_{22,x}}+2\underset{n-2}{a_{23}}+s\underset{n}{a_{23}}, \tag{2.26a}$$

$$2\underset{n-2}{a_{33}}+\bar{s}=-\underset{n-1}{a_{23,x}}+q\underset{n-1}{a_{12}}+2\underset{n-2}{a_{22}}+s, \quad \bar{q}\underset{n-1}{a_{11}}-2\underset{n-2}{a_{21}}=-(\frac{1}{2}\underset{n}{a_{23}}\underset{n-1}{a_{12}})_x+r\underset{n}{a_{23}}+q\underset{n-1}{a_{22}}, \tag{2.26b}$$

$$-\bar{q}\underset{n-1}{a_{12}}+2\underset{n-2}{a_{22}}+\bar{s}=2\underset{n-2}{a_{33}}+s. \tag{2.26c}$$

Proposition 2.1. *The matrix $\bar{U}$ determined by $\bar{U}=(T_x+TU)T^{-1}$ has the same form as U, that is,*

$$\bar{U}=\begin{pmatrix} 0 & -\bar{r}\lambda^{-1} & \bar{q} \\ \bar{r}\lambda^{-1} & 0 & -(2\lambda+\bar{s}\lambda^{-1}) \\ -\bar{q} & 2\lambda+\bar{s}\lambda^{-1} & 0 \end{pmatrix}, \tag{2.27}$$

where the transformation formulae from old potential into new ones are given by (2.25).

Proof. Define

$$(T_x+TU)T^*=\begin{pmatrix}\Theta_{11}(\lambda) & \Theta_{12}(\lambda) & \Theta_{13}(\lambda)\\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) & \Theta_{23}(\lambda)\\ \Theta_{31}(\lambda) & \Theta_{32}(\lambda) & \Theta_{33}(\lambda)\end{pmatrix} \tag{2.28}$$

and notice that $T^{-1}=\frac{T^*}{detT}$, We conclude that $\Theta_{ij}(\lambda)$ are $(3n+1)$-th-order polynomial in λ.

By means of (2.16) and (2.17), direct computation gives

$$\alpha_{j,x}^{(1)}=r\lambda^{-1}-(2\lambda+s\lambda^{-1})\alpha_j^{(2)}-(-r\lambda^{-1}\alpha_j^{(1)}+q\alpha_j^{(2)})\alpha_j^{(1)}, \tag{2.29a}$$

$$\alpha_{j,x}^{(2)}=-q+(2\lambda+s\lambda^{-1})\alpha_j^{(1)}-(-r\lambda^{-1}\alpha_j^{(1)}+q\alpha_j^{(2)})\alpha_j^{(2)}, \tag{2.29b}$$

Eq. (2.29) implies that λ_j $(1\leq j\leq 3n)$ are the roots of Θ_{ij} $(i,j=1,2,3)$ and Θ_{ij} may be divided by $detT$, which illustrates $(T_x+TU)T^{-1}$ is a first order polynomial in λ with matrix coefficients. Thus, we let

$$T_x+TU=(\bar{U}_1\lambda+\bar{U}_0+\bar{U}_{-1}\lambda^{-1})T \tag{2.30}$$

with the matrices $\bar{U}_1(x,t)$, $\bar{U}_0(x,t)$ and $\bar{U}_{-1}(x,t)$ independent of λ.

We denote $U=U_1\lambda+U_0+U_{-1}\lambda^{-1}$ with

$$U_1=\begin{pmatrix}0&0&0\\0&0&-2\\0&2&0\end{pmatrix},\quad U_0=\begin{pmatrix}0&0&q\\0&0&0\\-q&0&0\end{pmatrix},\quad U_{-1}=\begin{pmatrix}0&-r&0\\r&0&-s\\0&s&0\end{pmatrix}. \tag{2.31}$$

Via (2.25) and comparing the coefficients of λ^{n+1}, λ^n, λ^{n-1} lead to

- The $(n+1)$-th coefficients implies $T_nU_1=\bar{U}_1T_n$, from which we have

$$\bar{U}_1=\begin{pmatrix}0&0&0\\0&0&-2\\0&2&0\end{pmatrix}. \tag{2.32}$$

- The n-th coefficients implies $T_nU_0+T_{n-1}U_1+T_{n,x}=\bar{U}_1T_{n-1}+\bar{U}_0T_n$, from which we have

$$\bar{U}_0=\begin{pmatrix}0&0&\bar{q}\\0&0&0\\-\bar{q}&0&0\end{pmatrix}. \tag{2.33}$$

- The $(n-1)$-th coefficients implies $T_nU_{-1}+T_{n-2}U_1+T_{n-1}U_0+T_{n-1,x}=\bar{U}_1T_{n-2}+\bar{U}_0T_{n-1}+\bar{U}_{-1}T_n$, from which we have

$$\bar{U}_{-1}=\begin{pmatrix}0&-\bar{r}&0\\\bar{r}&0&-\bar{s}\\0&\bar{s}&0\end{pmatrix}. \tag{2.34}$$

We emphasize that $\bar{q}$, $\bar{r}$ and $\bar{s}$ shown here are defined by (2.25). The proof is completed. □

Proposition 2.2. *The matrix* $\bar{V}$ *in* (2.18) *has the same form as* V *under the transformation* (2.17) *and* (2.25).

Proof. Similar to Proposition 2.1, we can prove that $(T_x+TV)T^{-1}$ is a three-order polynomial in λ with matrix coefficients, that is

$$T_x+TV=(\bar{V}_3\lambda^3+\bar{V}_2\lambda^2+\bar{V}_1\lambda+\bar{V}_0)T, \tag{2.35}$$

with the matrices $\bar{V}_3$, $\bar{V}_2$, $\bar{V}_1$, and $\bar{V}_0$ independent of λ.

We denote $V=V_3\lambda^3+V_2\lambda^2+V_1\lambda+V_0$ with

$$V_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha(t) \\ 0 & \alpha(t) & 0 \end{pmatrix}, \quad V_2=\begin{pmatrix} 0 & 0 & \frac{\alpha}{2}q \\ 0 & 0 & 0 \\ -\frac{\alpha}{2}q & 0 & 0 \end{pmatrix}, \tag{2.36a}$$

$$V_1=\begin{pmatrix} 0 & -\left(\frac{\alpha}{2}r-\frac{\alpha}{4}q_x\right) & 0 \\ \frac{\alpha}{2}r-\frac{\alpha}{4}q_x & 0 & \frac{\alpha}{8}q^2 \\ 0 & -\frac{\alpha}{8}q^2 & 0 \end{pmatrix}, \tag{2.36b}$$

$$V_0=\begin{pmatrix} 0 & 0 & \frac{\alpha}{4}r_x-\frac{\alpha}{8}q_{xx}-\frac{\alpha}{16}q^3-\frac{\alpha}{4}qs \\ 0 & 0 & 0 \\ -\left(\frac{\alpha}{4}r_x-\frac{\alpha}{8}q_{xx}-\frac{\alpha}{16}q^3-\frac{\alpha}{4}qs\right) & 0 & 0 \end{pmatrix}. \tag{2.36c}$$

With the help of (2.25) and comparing the coefficients of λ^{n+3}, λ^{n+2}, λ^{n+1}, λ^n lead to

- The $(n+3)$-th coefficients implies $T_nV_3=\bar{V}_3T_n$, from which we have

$$\bar{V}_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha(t) \\ 0 & \alpha(t) & 0 \end{pmatrix}. \tag{2.37}$$

- The $(n+2)$-th coefficients implies $T_nV_2+T_{n-1}V_3=\bar{V}_3T_{n-1}+\bar{V}_2T_n$, from which we have

$$\bar{V}_2=\begin{pmatrix} 0 & 0 & \frac{\alpha}{2}\bar{q} \\ 0 & 0 & 0 \\ -\frac{\alpha}{2}\bar{q} & 0 & 0 \end{pmatrix}. \tag{2.38}$$

• The $(n+1)$-th coefficients implies $T_nV_1+T_{n-1}V_2+T_{n-2}V_3=\bar{V}_3T_{n-2}+\bar{V}_2T_{n-1}+\bar{V}_1T_n$, from which we have

$$\bar{V}_1=\begin{pmatrix} 0 & -\left(\frac{\alpha}{2}\bar{r}-\frac{\alpha}{4}\bar{q}_x\right) & 0 \\ \frac{\alpha}{2}\bar{r}-\frac{\alpha}{4}\bar{q}_x & 0 & \frac{\alpha}{8}\bar{q}^2 \\ 0 & -\frac{\alpha}{8}\bar{q}^2 & 0 \end{pmatrix}. \tag{2.39}$$

• The n-th coefficients implies $T_{n,x}+T_nV_0+T_{n-1}V_1+T_{n-2}V_2+T_{n-3}V_3=\bar{V}_3T_{n-3}+\bar{V}_2T_{n-2}+\bar{V}_1T_{n-1}+\bar{V}_0T_n$, from which we have

$$\bar{V}_0=\begin{pmatrix} 0 & 0 & \frac{\alpha}{4}\bar{r}_x-\frac{\alpha}{8}\bar{q}_{xx}-\frac{\alpha}{16}\bar{q}^3-\frac{\alpha}{4}\bar{q}\bar{s} \\ 0 & 0 & 0 \\ -\left(\frac{\alpha}{4}\bar{r}_x-\frac{\alpha}{8}\bar{q}_{xx}-\frac{\alpha}{16}\bar{q}^3-\frac{\alpha}{4}\bar{q}\bar{s}\right) & 0 & 0 \end{pmatrix}, \tag{2.40}$$

where $\bar{q}$ and $\bar{r}$ are defined in (2.25). This completes the proof. □

Propositions 2.1 and 2.2 show that both Lax pair (2.16) and (2.18) can lead to the same integrable system (2.8). Thus, we call the transformation (2.17) and (2.25) a Darboux transformation of the integrable system (2.8), by which the solution (q,r,s) of the integrable system (2.8) are mapped into its new solution $(\bar{q},\bar{r},\bar{s})$.

2.1.3 Nonclassical symmetries for (2.9)

There are large PDE systems for which it is difficult to determine their nonclassical symmetries due to the limited memory of the system on which the symbolic manipulation program runs. But recently, Nicoleta Bî1ă and Jitse Niesen proposed a new approach which overcomes the previous difficulties [33]. Starting from the PDE system, the standard procedure introduced by Bluman and Cole is analysed. The system is augmented with the invariant surface conditions, representing the characteristics of a fixed but arbitrary vector field X. Then the classical Lie method is applied to the augmented system. If this results in the same vector field X as is used to define the invariant surface conditions, then X is a nonclassical symmetry of the original PDE.

As an example, we are committed to searching for the nonclassical symmetries related to Eq. (2.9). In order to apply the nonclassical method, consider a one-parameter group of transformation generated by the following vector

$$Y=\xi(x,t,u)\partial_x+\eta(x,t,u)\partial_t+\phi(x,t,u)\partial_u.$$

If we seek symmetries with $\eta=1$, then the invariant surface condition

$$u_p^\alpha=-\sum_{j=1}^{p-1}\zeta^j(x,u)u_j^\alpha+\phi^\alpha(x,u)$$

turns into

$$u_t=\phi-\zeta u_x. \tag{2.41}$$

Substituting (2.41) into Eq. (2.9) results in

$$-\frac{\alpha}{8}u_{xxx}-\frac{3\alpha}{16}u^2u_x-\phi+\zeta u_x=0. \tag{2.42}$$

Meanwhile, the following relation

$$\begin{aligned}&\sum_{j=1}^{p-1}L^j\frac{\partial\Omega_k}{\partial x^j}+L^p\frac{\partial\Omega_k}{\partial x^p}+\sum_{\alpha=1}^{q}M^\alpha\frac{\partial\Omega_k}{\partial u^\alpha}\\&\quad+\sum_{\alpha=1}^{q}\sum_I\left[D_I\left(M^\alpha-\sum_{j=1}^{p-1}L^ju_j^\alpha\right)+\sum_{j=1}^{p-1}L^ju_{I,j}^\alpha\right]\frac{\partial\Omega_k}{\partial u_I^\alpha}\\&=\sum_{\alpha=1}^{q}\sum_I\left[D_I(L^pu_\alpha^p)-L^pu_{I,p}^\alpha\right]\frac{\partial\Omega_k}{\partial u_I^\alpha}\end{aligned} \tag{2.43}$$

can be simplified to

$$\begin{aligned}&L\frac{\partial\Omega}{\partial x}+K\frac{\partial\Omega}{\partial t}+M\frac{\partial\Omega}{\partial u}+(D_x(M-Lu_x)+Lu_{xx})\frac{\partial\Omega}{\partial u_x}\\&\quad+(D_x^3(M-Lu_x)+Lu_{xxxx})\frac{\partial\Omega}{\partial u_{xxx}}=0,\end{aligned} \tag{2.44}$$

where

$$\Omega=-\frac{\alpha}{8}u_{xxx}-\frac{3\alpha}{16}u^2u_x-\phi+\zeta u_x.$$

If we substitute $L=\zeta$, $K=1$ and $M=\phi$, $\alpha=16$ into (2.44), we get the nonlinear PDE system

$$\zeta_{uuu}=0,\quad 3\zeta_{xuu}-\phi_{uuu}=0, \tag{2.45a}$$

$$-\phi_t-\phi\phi_u-3u^2\phi_x-2\phi_{xxx}+6\zeta_x\phi=0,\quad -9\zeta\zeta_u+27\zeta_uu^2+6\zeta_{xuu}-6\phi_{xxu}=0, \tag{2.45b}$$

$$-6\phi_u+21\zeta_xu^2+9\zeta_u\phi-3\phi_uu^2+\zeta_t+\zeta\phi_u-4\phi_{xxu}+2\zeta_{xxx}-6\zeta\zeta_x=0, \tag{2.45c}$$

which represents the determining equations of the nonclassical symmetries associated with (2.9).

The general solution of the nonlinear PDE system (2.45) is

$$\zeta=c_1,\quad \phi=0, \tag{2.46}$$

where c_1 is a real number, and

$$\zeta=3u^2,\quad \phi=0. \tag{2.47}$$

As we can see, the case (2.46) corresponds to the classical operator $c_1X_1+X_2$. At this stage, we do not get all the classical operators, since we assumed that the coefficient of ∂_t is 1. We can retrieve all the classical operators by using that any mutiple of a nonclassical operator is again a nonclassical operator.

From (2.47) we get the nonclassical operator

$$Y=3u^2\partial_x+\partial_t.$$

2.2 Nonisospectral integrable hierarchy

Under the time evolution

$$\lambda_t=\sum_{m\geq 0}K_m(t)\lambda^{-m},$$

the resulting nonisospectral zero curvature equation

$$\vec{V}_x=\vec{U}\times\vec{V}+\frac{\partial\vec{U}}{\partial\lambda}\lambda_t \tag{2.48}$$

leads to

$$\begin{aligned}
&a_{m,x}=qc_m-rb_{m-1}+2K_m(t)-sK_{m-2}(t), &&(2.49a)\\
&b_{m,x}=-2c_{m+1}+ra_{m-1}-sc_{m-1}, &&(2.49b)\\
&c_{m,x}=2b_{m+1}-qa_m+sb_{m-1}-rK_{m-2}(t). &&(2.49c)
\end{aligned}$$

In terms of (2.49), one gets

$$\begin{aligned}
&b_0=c_0=0,\quad a_0=\alpha(t),\quad a_1=c_1=0,\quad b_1=\frac{\alpha q}{2}, &&(2.50a)\\
&b_2=0,\quad c_2=\frac{\alpha r}{2}-\frac{\alpha q_x}{4},\quad K_0=K_1=0,\quad K_2=\beta_2, &&(2.50b)\\
&a_2=-\frac{\alpha q^2}{8}+2\beta_2 x,\quad b_3=\frac{\alpha}{4}r_x-\frac{\alpha}{8}q_{xx}-\frac{\alpha}{16}q^3-\frac{\alpha}{4}qs+\beta_2 xq,\quad c_3=0,\quad a_3=0, &&(2.50c)\\
&c_4=-\frac{\alpha}{16}q^2r-\frac{\alpha}{4}rs+\frac{\alpha}{8}q_xs-\frac{\alpha}{8}r_{xx}+\frac{\alpha}{16}q_{xxx}+\frac{3\alpha}{32}q^2q_x+\frac{\alpha}{8}(qs)_x\\
&\qquad +\beta_2 xr-\frac{\beta_2 q}{2}-\frac{\beta_2 xq_x}{2},\cdots, &&(2.50d)\\
&a_{2m+1}=c_{2m+1}=0,\quad b_{2m+2}=0,\quad K_{2m+1}=0. &&(2.50e)
\end{aligned}$$

Denoting

$$\lambda_{t,+}^{(n)}=\sum_{m=0}^{n}K_m(t)\lambda^{n-m}=\lambda^n\lambda_t-\lambda_{t,-}^{(n)}=\lambda^n\lambda_t-\sum_{m=n+1}^{\infty}K_m(t)\lambda^{n-m},$$

then (2.48) can be broken down into

$$-\vec{V}_{+,x}^{(n)}+\vec{U}\times\vec{V}_{+}^{(n)}+\frac{\partial\vec{U}}{\partial\lambda}\lambda_{t,+}^{(n)}=(\vec{V}_{-}^{(n)})_x-\vec{U}\times\vec{V}_{-}^{(n)}-\frac{\partial\vec{U}}{\partial\lambda}\lambda_{t,-}^{(n)}.$$

By taking these terms which have the gradation 0, -1 and -2, we further get

$$\begin{aligned}&-\vec{V}_{+,x}^{(n)}+\vec{U}\times\vec{V}_{+}^{(n)}+\frac{\partial\vec{U}}{\partial\lambda}\lambda_{t,+}^{(n)}\\ =&(-rb_n-sK_{n-1})\vec{e}_1(-1)+(ra_n-sc_n)\vec{e}_2(-1)+(sb_n-rK_{n-1})\vec{e}_3(-1)\\ &-sK_n\vec{e}_1(-2)-rK_n\vec{e}_3(-2)-2b_{n+1}\vec{e}_3(0)+2c_{n+1})\vec{e}_2(0).\end{aligned}$$

Choose $n=2k+1$, one can find $\Delta_n=0$ so that $\vec{V}^{(n)}=\vec{V}_{+}^{(n)}$ the nonisospectral zero curvature equation

$$\frac{\partial\vec{U}}{\partial u}u_t+\frac{\partial\vec{U}}{\partial\lambda}\lambda_t-\vec{V}_x^{(n)}+\vec{U}\times\vec{V}^{(n)}=\vec{0} \tag{2.51}$$

leads to the nonisospectral mKdV integrable hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ s\end{pmatrix}_t=\begin{pmatrix} -2c_{2k+2}\\ -sb_{2k+1}+rK_{2k}\\ rb_{2k+1}+sK_{2k}\end{pmatrix}. \tag{2.52}$$

When $k=1$, the hierarchy (2.7) reduces to

$$\begin{aligned}q_t=&\frac{\alpha}{8}q^2r+\frac{\alpha}{2}rs-\frac{\alpha}{4}q_xs+\frac{\alpha}{4}r_{xx}-\frac{\alpha}{8}q_{xxx}-\frac{3\alpha}{16}q^2q_x\\ &-\frac{\alpha}{4}(qs)_x-2\beta_2xr+\beta_2q+\beta_2xq_x,\end{aligned} \tag{2.53a}$$

$$r_t=-\frac{\alpha}{4}r_xs+\frac{\alpha}{8}q_{xx}s+\frac{\alpha}{16}q^3s+\frac{\alpha}{4}qs^2-\beta_2xqs+\beta_2r, \tag{2.53b}$$

$$s_t=\frac{\alpha}{4}rr_x-\frac{\alpha}{8}q_{xx}r-\frac{\alpha}{16}q^3r-\frac{\alpha}{4}qsr+\beta_2xqr+\beta_2s. \tag{2.53c}$$

Taking $r=s=0$, (2.53) becomes

$$q_t=-\frac{\alpha}{8}q_{xxx}-\frac{3\alpha}{16}q^2q_x+\beta_2xq_x+\beta_2q, \tag{2.54}$$

which is nonisospectral mKdV equation.

2.2.1 Covering and nonlocal symmetries for (2.54)

In this section, we consider a differential equation $\mathscr{Y}$ and its covering $\tilde{\mathscr{Y}}_\infty$. In [34, 35], Krasilshckik and Vinogradov introduced a direct and effective technique, which overcomes the shortcomings in [36], to construct nonlocal symmetries of a system $\mathscr{Y}$ of differential equations. It turns out that the nonlocal symmetries of differential equations are determined by their generating functions, which are intego-differential type operators.

Given an infinitely-dimensional manifold $\tilde{\mathscr{Y}}_\infty$ equipped with an n-dimensional integrable distribution. This implies that in the tangent space $Y_y(\tilde{\mathscr{Y}}_\infty)$, which is infinitely-dimensional, at any point $y\in\tilde{\mathscr{Y}}_\infty$, an n-dimensional subspace $\tilde{\mathscr{L}}_y\in T_y(\tilde{\mathscr{Y}}_\infty)$ is determined and that the system $\{\tilde{\mathscr{L}}_y\}$ of these subspaces satisfies the condition of the classical Frobenius theorem. At the same time, if there is a regular map τ from $\tilde{\mathscr{Y}}_\infty$ onto $\mathscr{Y}_\infty$ introducing an isomorphism of $\tilde{\mathscr{L}}_y$ and $\mathscr{L}_{\tau(y)}\in T_{\tau(y)}(\mathscr{Y}_\infty)$, then we say that $\tilde{\mathscr{Y}}_\infty$ is the covering of the equation $\mathscr{Y}$. This means that any integral manifold $U\in\tilde{\mathscr{Y}}_\infty$ (i.e., such a manifold that $T_y(U)=\tilde{\mathscr{L}}_y,\forall y\in U$) is mapped by τ onto an integrable manifold $V=\tau(U)\in\mathscr{Y}_\infty$; that is onto a solution of the equation $\mathscr{Y}$.

We firstly illustrate the related coordinates and notations. Let $W\in\mathbb{R}^N$, $0<N\leq\infty$, be a domain in $\mathbb{R}^n$ and $w_1,w_2,\cdots$ be the standard coordinates in W. Every manifold $\tilde{\mathscr{Y}}_\infty$ can be represented locally as the Cartesian product $\tilde{\mathscr{Y}}_\infty=\mathscr{Y}_\infty\times W$ and the mapping τ is defined by the natural projection from $\mathscr{Y}_\infty\times W$ to $\mathscr{Y}_\infty$. It is introduced in [34] that a transformation $f:\tilde{\mathscr{Y}}_\infty\rightarrow\mathscr{Y}_\infty$ is said to be a nonlocal symmetry of $\mathscr{Y}$ if and only if it preserves the contact structure on $\tilde{\mathscr{Y}}_\infty$. In other words, f is a nonlocal symmetry when $f_*(\tilde{\mathscr{L}}_y)=\tilde{\mathscr{L}}_{f(y)}$ for any point $\mathscr{Y}\in\tilde{\mathscr{Y}}_\infty$. The contact structure on $\mathscr{Y}_\infty$ is determined by system consisting of n vector fields $\bar{D}_i$ and $\bar{D}_i$ is the restriction of the total derivative operator on $\mathscr{Y}_\infty$. Furthermore, the equalities $[\bar{D}_i,\bar{D}_j]=0$ hold. Then an n-dimensional contact structure on $\tilde{\mathscr{Y}}_\infty=\mathscr{Y}_\infty\times W$ may be determined by a system of vector fields $\tilde{D}_i=\bar{D}_i+X_i$, $i=1,\cdots,n$, where

$$X_i=\sum_j X_{ij}\frac{\partial}{\partial w_j},\quad X_{ij}\in C^\infty(\tilde{\mathscr{Y}}_\infty).$$

Thus, the Frobenius conditions can now be replaced by $[\tilde{D}_i,\tilde{D}_j]=0$, which can be expressed in the following equivalent form

$$[\bar{D}_i,X_j]+[X_i,\bar{D}_j]+[X_i,X_j]=0. \tag{2.55}$$

Relation (2.55) describes all the coverings of the equation $\mathscr{Y}$ with the fibre W, where the coordinates w_i and the operators $\tilde{D}_i$ are regarded as "nonlocal variables" and "total derivatives", respectively.

The Lie factor algebra

$$Sym_\tau \mathscr{Y} = \frac{D_{\mathscr{L}}(\tilde{\mathscr{Y}}_\infty)}{\mathscr{L}D(\tilde{\mathscr{Y}}_\infty)} \tag{2.56}$$

is said to be the algebra of nonlocal symmetries of the type τ for the equation $\mathscr{Y}$. Here $D_{\mathscr{L}}(\tilde{\mathscr{Y}}_\infty)$ consists of such fields S on $\tilde{\mathscr{Y}}_\infty$ that $[S,\mathscr{L}D(\tilde{\mathscr{Y}}_\infty)] \in \mathscr{L}D(\tilde{\mathscr{Y}}_\infty)$, while

$$\mathscr{L}D(\tilde{\mathscr{Y}}_\infty) = \left\{ \sum_{i=1}^{n} \varphi_i \tilde{D}_i \,|\, \varphi_i \in \mathscr{L}^\infty(\tilde{\mathscr{Y}}_\infty) \right\}.$$

The elements of $Sym_\tau \mathscr{Y}$ can be identified with vector yields S on $\tilde{\mathscr{Y}}_\infty$, such that

$$[S,\tilde{D}_i] = 0, \quad i = 1,\cdots,n, \tag{2.57}$$

and $S(\varphi) = 0$, for any $\varphi = \varphi(x)$.

Now we return to the nonisospectral MKdV equation

$$\mathscr{Y} = \left\{ u_t = -\frac{\alpha}{8} u_{xxx} - \frac{3\alpha}{16} u^2 u_x + \beta_2 x u_x + \beta_2 u \right\}. \tag{2.58}$$

Taking $\alpha = 16$, (2.58) can be we rewritten as the following form

$$p_2 = -2p_{(3)} - 3p_{(0)}^2 p_{(1)} + \beta_2 x p_{(1)} + \beta_2 p_{(0)}. \tag{2.59}$$

Denote $\tilde{D}_x = D_x + X$, $\tilde{D}_t = D_t + T$, where

$$X = \sum_{i\geq 1} X_i \frac{\partial}{\partial w_i}, \quad T = \sum_{j\geq 1} T_j \frac{\partial}{\partial w_j}, \quad X_i, T_j \in \mathscr{L}^\infty(\tilde{\mathscr{Y}}_\infty).$$

Thus, (2.55) can be transformed into

$$[D_x,T] + [X,D_t] + [X,T] = 0, \tag{2.60}$$

where

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + p_{(1)} \frac{\partial}{\partial p_{(0)}} + p_{(2)} \frac{\partial}{\partial p_{(1)}} + \cdots + p_{(k+1)} \frac{\partial}{\partial p_{(k)}} + \cdots, \\
D_t &= \frac{\partial}{\partial t} + \left(-\frac{\alpha}{8} p_{(3)} - \frac{3\alpha}{16} p_{(0)}^2 p_{(1)} + \beta_2 x p_{(1)} + \beta_2 p_{(0)} \right) \frac{\partial}{\partial p_{(0)}} \\
&\quad + D_x \left(-\frac{\alpha}{8} p_{(3)} - \frac{3\alpha}{16} p_{(0)}^2 p_{(1)} + \beta_2 x p_{(1)} + \beta_2 p_{(0)} \right) \frac{\partial}{\partial p_{(1)}} + \cdots \\
&\quad + D_x^k \left(-\frac{\alpha}{8} p_{(3)} - \frac{3\alpha}{16} p_{(0)}^2 p_{(1)} + \beta_2 x p_{(1)} + \beta_2 p_{(0)} \right) \frac{\partial}{\partial p_{(k)}} + \cdots.
\end{aligned}$$

We shall find all the solution for which the function X_i, T_j only depend on the variables $p_{(0)}$, $p_{(1)}$, $p_{(2)}$. A direct calculation gives

$$\begin{aligned}&p_{(1)}\frac{\partial T}{\partial p_{(0)}}+p_{(2)}\frac{\partial T}{\partial p_{(1)}}+p_{(3)}\frac{\partial T}{\partial p_{(2)}}-\big(-2p_{(3)}-3p_{(0)}^2p_{(1)}+\beta_2p_{(0)}+\beta_2xp_{(1)}\big)\frac{\partial X}{\partial p_{(0)}}\\&\quad-\big(-2p_{(4)}-3p_{(0)}^2p_{(2)}-6p_{(1)}^2p_{(0)}+\beta_2p_{(1)}+\beta_2xp_{(2)}\big)\frac{\partial X}{\partial p_{(1)}}\\&\quad-\big(-2p_{(5)}-6p_{(0)}p_{(1)}p_{(2)}-3p_{(0)}^2p_{(3)}-6p_{(1)}^3-12p_{(0)}p_{(1)}p_{(2)}+\beta_2p_{(2)}+\beta_2xp_{(3)}\big)\frac{\partial X}{\partial p_{(2)}}\\&\quad+[X,T]=0.\end{aligned} \tag{2.61}$$

Setting the coefficients of $p_{(5)}$, $p_{(4)}$, $p_{(3)}$ to be zero, we have

$$\frac{\partial X}{\partial p_{(2)}}=0,\quad \frac{\partial X}{\partial p_{(1)}}=0,\quad \frac{\partial T}{\partial p_{(2)}}+2\frac{\partial X}{\partial p_{(0)}}=0, \tag{2.62}$$

which implies that the function X do not depend on $p_{(1)}$, $p_{(2)}$ and

$$T=-2p_{(2)}\frac{\partial X}{\partial p_{(0)}}+R(p_{(0)},p_{(1)}).$$

Substituting the derived results into (2.61) yields

$$\begin{aligned}&-2p_{(1)}p_{(2)}\frac{\partial^2 X}{\partial p_{(0)}^2}+p_{(1)}\frac{\partial R}{\partial p_{(0)}}+p_{(2)}\frac{\partial R}{\partial p_{(1)}}-\big(-3p_{(0)}^2p_{(1)}+\beta_2p_{(0)}+\beta_2xp_{(1)}\big)\frac{\partial X}{\partial p_{(0)}}\\&\quad+\left[X,-2p_{(2)}\frac{\partial X}{\partial p_{(0)}}+R\right]=0.\end{aligned} \tag{2.63}$$

Assume that $\frac{\partial R}{\partial p_{(1)}}$ does not depend on $p_{(1)}$, then the coefficient of $p_{(1)}p_{(2)}$ vanishes, that is

$$\frac{\partial^2 X}{\partial p_{(0)}^2}=0 \quad\Rightarrow\quad X=p_{(0)}A+B, \tag{2.64}$$

where the coefficients of the field A and B depend on the variables w_i only.

Substituting (2.64) into (2.63) yields

$$\begin{aligned}&p_{(1)}\frac{\partial R}{\partial p_{(0)}}+p_{(2)}\frac{\partial R}{\partial p_{(1)}}-\big(-3p_{(0)}^2p_{(1)}+\beta_2p_{(0)}+\beta_2xp_{(1)}\big)A\\&\quad+\big[p_{(0)}A+B,-2p_{(2)}A+R\big]=0.\end{aligned} \tag{2.65}$$

Equating the coefficients of $p_{(2)}$ to zero, we get

$$\frac{\partial R}{\partial p_{(1)}}-2[B,A]=0 \quad \Rightarrow \quad R=2p_{(1)}[B,A]+S(p_{(0)}), \tag{2.66}$$

substituting it into (2.65) leads to

$$p_{(1)}\frac{\partial S}{\partial p_{(0)}}-\left(-3p_{(0)}^2p_{(1)}+\beta_2p_{(0)}+\beta_2xp_{(1)}\right)A+\left[p_{(0)}A+B,2p_{(1)}[B,A]+S\right]=0. \tag{2.67}$$

As S do not depend on $p_{(1)}$, the equation is equivalent to the following system

$$\frac{\partial S}{\partial p_{(0)}}+3p_{(0)}^2A-\beta_2xA+\left[p_{(0)}A+B,2[B,A]\right]=0, \quad -\beta_2p_{(0)}A+\left[p_{(0)}A+B,S\right]. \tag{2.68}$$

It follows from the first equation that

$$S=-p_{(0)}^3A+\beta_2xAp_{(0)}-p_{(0)}^2[A,[B,A]]-2p_{(0)}[B,[B,A]]+C. \tag{2.69}$$

Meanwhile, the second equation in (2.68) transforms into

$$\begin{aligned}&-\beta_2p_{(0)}A-p_{(0)}^3[A,[A,[B,A]]]-2p_{(0)}^2[A,[B,[B,A]]]+p_{(0)}[A,C]-p_{(0)}^3[B,A]\\&\quad+\beta_2xp_{(0)}[B,A]-p_{(0)}^2[B,[A,[B,A]]]-2p_{(0)}[B,[B,[B,A]]]+[B,C]=0.\end{aligned} \tag{2.70}$$

Equating the coefficients of all powers of $p_{(0)}$ to zero, we get the system in the following

$$[A,[A,[B,A]]]+[B,A]=0, \quad [A,[B,[B,A]]]=0, \tag{2.71a}$$

$$-\beta_2A+[A,C]+\beta_2x[B,A]-2[B,[B,[B,A]]]=0, \quad [B,C]=0. \tag{2.71b}$$

Thus, one has

Theorem 2.1. *Any covering of the nonisospectral mKdV equation in which the coefficients of the fields X and T only depend on $p_{(0)}$, $p_{(1)}$, $p_{(2)}$, are determined by the field of the form*

$$\tilde{D}_x=D_x+p_{(0)}A+B, \tag{2.72a}$$

$$\begin{aligned}\tilde{D}_t=&D_t-2p_{(2)}A+2p_{(1)}[B,A]-p_{(0)}^3A+\beta_2xAp_{(0)}-p_{(0)}^2[A,[B,A]]\\&-2p_{(0)}[B,[B,A]]+C,\end{aligned} \tag{2.72b}$$

where A, B, C satisfy (2.71).

Subsequently, we focus on calculating the nonlocal symmetries of (2.54) with the aid of the covering described in Theorem 1.1. We identity the elements of $Sym_\tau\mathscr{Y}$ with fields S on $\tilde{\mathscr{Y}}_\infty$, such that $S=P+\Phi$, where

$$P=\sum_{i\geq 0}P_i\frac{\partial}{\partial p_{(i)}},\quad \Phi=\sum_{j\geq 0}\Phi_j\frac{\partial}{\partial w_j}\quad \text{and}\quad [S,\tilde{D}_x]=[S,\tilde{D}_t]=0.$$

Taking (2.72) into consideration, one gets

$$[S,\tilde{D}_x]=\sum_i(P_{i+1}-\tilde{D}_x(P_i))\frac{\partial}{\partial p_{(i)}}+P_0A+[\Phi,\tilde{D}_x]=0, \tag{2.73a}$$

$$\begin{aligned}[S,\tilde{D}_t]=&\sum_i\left(\sum_k P_k\frac{\partial}{\partial p_{(k)}}\left(\tilde{D}_x^k\left(-2p_{(3)}-3p_{(0)}^2p_{(1)}+\beta_2xp_{(1)}+\beta_2p_{(0)}\right)\right)-\tilde{D}_t(p_{(i)})\right)\frac{\partial}{\partial p_{(i)}}\\&-2P_2A+2P_1[B,A]-3p_{(0)}^2P_0A+\beta_2xAP_0-2p_{(0)}P_0[A,[B,A]]\\&-2P_0[B,[B,A]]+[\Phi,\tilde{D}_t]=0.\end{aligned} \tag{2.73b}$$

In terms of (2.73a), we get

$$P_{i+1}=\tilde{D}_x(P_i),\quad i=0,1,\cdots,\quad \Rightarrow\quad P_i=\tilde{D}_x^i(\psi),\psi=P_0, \tag{2.74a}$$

$$P_0A+[\Phi,\tilde{D}_x]=0. \tag{2.74b}$$

Denoting

$$\tilde{\varepsilon}_\psi=\sum_k\tilde{D}_x^k(\psi)\frac{\partial}{\partial p_{(k)}},$$

then (2.73b) yields

$$\left(\tilde{\varepsilon}_\psi\circ\tilde{D}_x^i\right)\left(-2p_{(3)}-3p_{(0)}^2p_{(1)}+\beta_2xp_{(1)}+\beta_2p_{(0)}\right)=\left(\tilde{D}_t\circ\tilde{D}_x^i\right)(\psi), \tag{2.75a}$$

$$\begin{aligned}&[\Phi,\tilde{D}_t]-2P_2A+2P_1[B,A]-3p_{(0)}^2P_0A+\beta_2xAP_0\\&\quad-2p_{(0)}P_0[A,[B,A]-2P_0[B,[B,A]=0.\end{aligned} \tag{2.75b}$$

From (2.75a), we get

$$\tilde{\varepsilon}_\psi(-2p_{(3)}-3p_{(0)}^2p_{(1)}+\beta_2xp_{(1)}+\beta_2p_{(0)})=\tilde{D}_t(\psi),$$

which is equivalent to

$$-2\tilde{D}_x^3(\psi)-3p_{(0)}^2\tilde{D}_x(\psi)-6p_{(0)}p_{(1)}\psi+\beta_2\psi+\beta_2x\tilde{D}_x(\psi)=\tilde{D}_t(\psi). \tag{2.76}$$

Note that (2.75b) can be rewritten as $\tilde{l}_F(\psi)=0$, when

$$F=-2p_{(3)}-3p_{(0)}^2p_{(1)}+\beta_2xp_{(1)}+\beta_2p_{(0)}-p_2,$$

which is nonisospectral mKdV equation.

Proposition 2.3. *Any nonlocal symmetry of Burgers' equation in the covering (2.72) is of the form* $S=\tilde{\varepsilon}_\psi+\Phi$*, where*

$$\Phi=\sum_i \Phi_i\left(\frac{\partial}{\partial w_i}\right),\quad \Phi_i\in C^\infty(\mathscr{Y}_\infty),$$

Φ *and* ψ *satisfy the system of the following differential equations*

$$\begin{cases}\psi A=[\tilde{D}_x,\Phi]-2\tilde{D}_x^2(\psi)A+2\tilde{D}_x(\psi)[B,A]-3p_{(0)}^2\psi A+\beta_2 A\psi\\ \qquad -2p_{(0)}[A,[B,A]\psi-2[B,[B,A]\psi=[\tilde{D}_t,\Phi],\\ \tilde{l}_F=-2\tilde{D}_x^3(\psi)-3p_{(0)}^2\tilde{D}_x(\psi)-6p_{(0)}p_{(1)}\psi+\beta_2\psi+\beta_2 x\tilde{D}_x(\psi)=\tilde{D}_t(\psi)=0.\end{cases}\tag{2.77}$$

When $A=B=C=0$, the covering is trivial and the first two equation in (2.77) are satisfied in a trivial way. As a consequence, this system $\tilde{l}_F=0$. Thus, the local theory of symmetries is a natural part of the nonlocal one.

Now, let's talk about A,B and C in a few cases.

(**I**): $A=0$, $B=\lambda\frac{\partial}{\partial w}$, $C=\mu\frac{\partial}{\partial w}$, $\lambda,\mu=const$. In this case we have

$$\tilde{D}_x=D_x+\lambda\frac{\partial}{\partial w},\quad \tilde{D}_t=D_t+\mu\frac{\partial}{\partial w}.$$

Suppose $v=w-\lambda x-\mu t$, then $\tilde{D}_x(v)=\tilde{D}_t(v)=0$. Hence, in $v,x,t\cdots,p_{(k)},\cdots$, coordinate system the equalities $\tilde{D}_x=D_x$, $\tilde{D}_t=D_t$ hold.

(**II**): $A=\frac{\partial}{\partial w}$, $B=b\frac{\partial}{\partial w}$, $C=c\frac{\partial}{\partial w}$, $b=c=const$. In accordance to (2.72), we have

$$\begin{aligned}&\tilde{D}_x=D_x+(p_{(0)}+b)\frac{\partial}{\partial w},\\ &\tilde{D}_t=D_t+(-2p_{(2)}-p_{(0)}^3+\beta_2 xp_{(0)}+c)\frac{\partial}{\partial w}.\end{aligned}$$

Using the coordinate $v=w-bx-ct$ instead of w, we get

$$\begin{aligned}&\tilde{D}_x=D_x+p_{(0)}\frac{\partial}{\partial w},\\ &\tilde{D}_t=D_t+(-2p_{(2)}-p_{(0)}^3+\beta_2 xp_{(0)})\frac{\partial}{\partial w}.\end{aligned}$$

(**III**): $A=\frac{\partial}{\partial w}$, $B=0$, $C=0$. In this case (2.77) transforms into

$$\psi=\tilde{D}_x(\varphi),\quad -2\tilde{D}_x^2(\psi)-3p_{(0)}^2\psi+\beta_2 x\psi=\tilde{D}_t(\varphi),\quad \tilde{l}_F(\psi)=0,\tag{2.78}$$

where $\Phi=\varphi(\frac{\partial}{\partial w})$. Note that the third system in (2.78) is the consequence of the first two and so it can be transformed into

$$-2\tilde{D}_x^3(\varphi)-3p_{(0)}^2\tilde{D}_x(\varphi)+\beta_2 x\tilde{D}_x(\varphi)=\tilde{D}_t(\varphi). \tag{2.79}$$

Denoting $v=p_{(-1)}$, then the second the relations (2.79) is of the form $l_G(\varphi)=0$, where $G=-2v_{xxx}-v_x^3+\beta_2 xv_x-v_t$. This gives an idea that the manifold $\tilde{\mathscr{Y}}_\infty$ is of the form $\mathscr{Y}'_\infty$, where $\mathscr{Y}'$ is the equation $v_t=-2v_{xxx}-v_x^3+\beta_2 xv_x$. In fact, we can identity $\mathscr{Y}'_\infty$ and $\tilde{\mathscr{Y}}_\infty$ together with respective contact structures by patting $x=x'$, $t=t'$, $p_{(k)}=p'_{(k+1)}$, $v=p'_{(0)}$, where x', t',$p'_{(k)}$ are the standard coordinates on $\mathscr{Y}'_\infty$. Then it follows that $Sym_\tau\mathscr{Y}=Sym\mathscr{Y}'$. The algebra $Sym\mathscr{Y}'$ can be calculated by the same scheme as was used in [36].

(IV): $A=a(w)\frac{\partial}{\partial w}$, $B=0$, $C=0$, then in the case considered, (2.77) transforms into

$$\begin{aligned}\psi=&\frac{1}{a}(\tilde{D}_x(\varphi)-p_{(0)}a'\varphi)-2\tilde{D}_x^3(\varphi)+6p_{(0)}a'\tilde{D}_x^2(\varphi)+(6p_{(1)}a'-6p_{(0)}^2a'^2+6p_{(0)}^2a''a\\&-3p_{(0)}^2+\beta_2 x)\tilde{D}_x(\varphi)+(4p_{(0)}p_{(1)}a''a-6p_{(0)}p_{(1)}a'^2-4p_{(0)}^3aa'a''+2p_{(0)}^3a^2a'''\\&+2p_{(0)}^3a'^3+2p_{(0)}^3a')\varphi=\tilde{D}_t(\varphi).\end{aligned} \tag{2.80}$$

When $a'=1$, the second equation in this system (2.80) reduces to

$$\begin{aligned}&-2\tilde{D}_x^3(\varphi)+6p_{(0)}\tilde{D}_x^2(\varphi)+(6p_{(1)}-9p_{(0)}^2+\beta_2 x)\tilde{D}_x(\varphi)\\&\quad+(-6p_{(0)}p_{(1)}+4p_{(0)}^3)\varphi=\tilde{D}_t(\varphi),\end{aligned} \tag{2.81}$$

which is also the determined equation for the nonlocal symmetries of (2.58).

3 A high-dimensional vector-product Lie algebra and its application

In this section, we enlarge the vector-product Lie algebra $\mathcal{S}$ to a six-dimensional case. In order to conveniently write, we omit the symbol "$\rightarrow$" upper every vector, for example, the vector $\vec{E_i}$ is simply written as E_i $(i=1,2,\cdots)$. Denote by

$$G=span\{E_1,E_2,E_3,E_4,E_5,E_6\}, \tag{3.1}$$

which is a Lie algebra along with the following vector-product relations

$$\begin{aligned}&E_1\times E_2=E_3, && E_2\times E_3=E_1, && E_3\times E_1=E_2, && E_1\times E_4=E_5, && E_1\times E_5=-E_4,\\&E_1\times E_6=0, && E_2\times E_4=0, && E_2\times E_5=E_6, && E_2\times E_6=-E_5, && E_3\times E_4=-E_6,\\&E_3\times E_5=0, && E_3\times E_6=E_4, && E_4\times E_5=E_1, && E_4\times E_6=-E_3, && E_5\times E_6=E_2.\end{aligned}$$

Assume that

$$G=G_1\oplus G_2,\quad G_1=span\{E_1,E_2,E_3\},\quad G_2=span\{E_4,E_5,E_6\},$$

then we find that

$$G_1\cong\mathcal{S},G_1\times G_2\subset G_2,\quad G_2\times G_2\subset G_1,$$

which implies that the Lie algebra is a symmetric Lie algebra of reductive homogeneous space. It is remarkable that the relation has Lie bracket computation, while here G possesses the vector products.

In the following section, we aim to introduce the linear spectral problem with the aid of two loop algebras corresponding to the Lie algebra G, The first vector-product loop algebra is given by

$$\tilde{G}=span\{E_1(n),E_2(n),E_3(n),E_4(n),E_5(n),E_6(n)\},\tag{3.2}$$

of which $E_i(n)=E_i\otimes\lambda^n$, $n\in N$, $i=1,2,\cdots$, $degE_i(n)=n$, equipped with the following operation relations

$$\begin{aligned}
&E_1(n)\times E_2(m)=E_3(m+n), && E_2(m)\times E_3(n)=E_1(m+n),\\
&E_3(m)\times E_1(n)=E_2(m+n), && E_1(m)\times E_5(n)=-E_4(m+n),\\
&E_1(m)\times E_6(n)=0, && E_2(m)\times E_4(n)=0,\\
&E_2(m)\times E_5(n)=E_6(m+n), && E_2(m)\times E_6(n)=-E_5(m+n),\\
&E_3(m)\times E_4(n)=-E_6(m+n), && E_3(m)\times E_5(n)=0,\\
&E_3(m)\times E_6(n)=E_4(m+n), && E_4(m)\times E_5(n)=E_1(m+n),\\
&E_4(m)\times E_6(n)=-E_3(m+n), && E_4(m)\times E_6(n)=-E_3(m+n),\\
&E_1(m)\times E_4(n)=E_5(m+n), && E_5(m)\times E_6(n)=E_2(m+n).
\end{aligned}$$

The first vector-product loop algebra is applied to introduce the following spectral problem

$$\vec{\varphi}_x=\vec{U}\times\vec{\varphi},\tag{3.3a}$$

$$\begin{aligned}\vec{U}=&2E_1(1)+qE_2(0)+rE_3(-1)+sE_1(-1)\\&+u_1E_4(-1)+u_2E_5(0),\end{aligned}\tag{3.3b}$$

$$\vec{\varphi}_t=\vec{V}\times\vec{\varphi},\tag{3.3c}$$

where

$$\begin{aligned}
\vec{V}=&\vec{V}_1+\vec{V}_2,\\
\vec{V}_1=&\sum_{i\geq 0}\theta_{1,i}E_1(-i)+\theta_{2,i}E_2(-i)+\theta_{3,i}E_3(-i)+\theta_{4,i}E_4(-i)\\
&\quad+\theta_{5,i}E_5(-i)+\theta_{6,i}E_6(-i),\\
\vec{V}_2=&\sum_{j\geq 0}\bar{\theta}_{1,j}E_1(-j)+\bar{\theta}_{2,j}E_2(-j)+\bar{\theta}_{3,j}E_3(-j)+\bar{\theta}_{4,j}E_4(-j)\\
&\quad+\bar{\theta}_{5,j}E_5(-j)+\bar{\theta}_{6,j}E_6(-j),\\
\lambda_t=&\frac{\partial\lambda}{\partial t}=\sum_{j\geq 0}K_j(t)\lambda^{-j}.
\end{aligned}$$

The compatibility condition of (3.3) presents

$$\frac{\partial\vec{U}}{\partial u}u_t+\frac{\partial\vec{U}}{\partial\lambda}\lambda_t-\vec{V}_x+\vec{U}\times\vec{V}=\vec{0}, \tag{3.4}$$

which can be decomposed into the following form

$$\frac{\partial\vec{U}}{\partial u}u_t+\frac{\partial\vec{U}}{\partial\lambda}\lambda_t-\vec{V}_{1,x}+\vec{U}\times\vec{V}_1-\vec{V}_{2,x}+\vec{U}\times\vec{V}_2=\vec{0}. \tag{3.5}$$

Then the stationary zero curvature equation

$$\vec{V}_{1,x}=\vec{U}\times\vec{V}_1 \tag{3.6}$$

generates

$$\begin{aligned}
&\theta_{1,i,x}=q\theta_{3,i}-r\theta_{2,i-1}+u_1\theta_{5,i-1}-u_2\theta_{4,i}, &&\text{(3.7a)}\\
&\theta_{2,i,x}=-2\theta_{3,i+1}+r\theta_{1,i-1}-s\theta_{3,i-1}+u_2\theta_{6,i}, &&\text{(3.7b)}\\
&\theta_{3,i,x}=2\theta_{2,i+1}-q\theta_{1,i}+s\theta_{2,i-1}-u_1\theta_{6,i-1}, &&\text{(3.7c)}\\
&\theta_{4,i,x}=-2\theta_{5,i+1}+r\theta_{6,i-1}-s\theta_{5,i-1}+u_2\theta_{1,i}, &&\text{(3.7d)}\\
&\theta_{5,i,x}=2\theta_{4,i+1}-q\theta_{6,i}+s\theta_{4,i-1}-u_1\theta_{1,i-1}, &&\text{(3.7e)}\\
&\theta_{6,i,x}=q\theta_{5,i}-r\theta_{4,i-1}+u_1\theta_{3,i-1}-u_2\theta_{2,i}. &&\text{(3.7f)}
\end{aligned}$$

Under the time evolution $\lambda_t\neq 0$, the resulting zero curvature equation reads

$$\vec{V}_{2,x}=\vec{U}\times\vec{V}_2+\frac{\partial\vec{U}}{\partial\lambda}\lambda_t, \tag{3.8}$$

which leads to

$$\bar{\theta}_{1,j,x}=q\bar{\theta}_{3,j}-r\bar{\theta}_{2,j-1}+u_1\bar{\theta}_{5,j-1}-u_2\bar{\theta}_{4,j}+2K_j-sK_{j-2}, \tag{3.9a}$$
$$\bar{\theta}_{2,j,x}=-2\bar{\theta}_{3,j+1}+r\bar{\theta}_{1,j-1}-s\bar{\theta}_{3,j-1}+u_2\bar{\theta}_{6,j}, \tag{3.9b}$$
$$\bar{\theta}_{3,j,x}=2\bar{\theta}_{2,j+1}-q\bar{\theta}_{1,j}+s\bar{\theta}_{2,j-1}-u_1\bar{\theta}_{6,j-1}-rK_{j-2}, \tag{3.9c}$$
$$\bar{\theta}_{4,j,x}=-2\bar{\theta}_{5,j+1}+r\bar{\theta}_{6,j-1}-s\bar{\theta}_{5,j-1}+u_2\bar{\theta}_{1,j}-u_1K_{j-2}, \tag{3.9d}$$
$$\bar{\theta}_{5,j,x}=2\bar{\theta}_{4,j+1}-q\bar{\theta}_{6,j}+s\bar{\theta}_{4,j-1}-u_1\bar{\theta}_{1,j-1}, \tag{3.9e}$$
$$\bar{\theta}_{6,j,x}=q\bar{\theta}_{5,j}-r\bar{\theta}_{4,j-1}+u_1\bar{\theta}_{3,j-1}-u_2\bar{\theta}_{2,j}. \tag{3.9f}$$

In terms of (3.7) and (3.9), we take initial values

$$\begin{aligned}&\theta_{2,0}=\theta_{3,0}=\theta_{4,0}=\theta_{5,0}=\theta_{6,0}=0, \qquad \theta_{1,0}=\alpha(t),\\ &\bar{\theta}_{2,0}=\bar{\theta}_{3,0}=\bar{\theta}_{5,0}=\bar{\theta}_{4,0}=0, \quad \bar{\theta}_{1,0}=\bar{\alpha}(t), \quad K_2=\beta_2(t).\end{aligned}$$

Then one has

$$\begin{aligned}
&\theta_{2,1}=\frac{\alpha}{2}q, \quad \theta_{5,1}=\frac{\alpha}{2}u_2, \quad \theta_{3,1}=0, \quad \theta_{1,1}=0, \quad \theta_{2,2}=0,\\
&\theta_{4,2}=\frac{\alpha}{2}u_{2x}+\frac{\alpha}{2}u_1, \quad \theta_{3,2}=-\frac{\alpha}{4}q_x+\frac{\alpha r}{2}, \quad \theta_{1,2}=-\frac{\alpha}{4}q^2-\frac{\alpha}{4}u_2^2,\\
&\theta_{2,3}=-\frac{\alpha}{4}q_{xx}+\frac{\alpha}{4}r_x-\frac{\alpha}{8}q^3-\frac{\alpha}{8}qu_2^2-\frac{\alpha}{4}qs, \quad \theta_{5,3}=-\frac{\alpha}{8}q^2u_2-\frac{\alpha}{8}u_2^3-\frac{\alpha}{4}u_2s-\frac{\alpha}{4}u_{2,xx}-\frac{\alpha}{4}u_{1x},\\
&\theta_{6,3}=-\frac{\alpha}{4}qu_1-\frac{\alpha}{4}u_2r+\int\Theta dx, \quad \Theta=\frac{\alpha}{4}q_{xx}u_2-\frac{\alpha}{4}u_{2,xx}q-\frac{\alpha}{4}q_xu_1-\frac{\alpha}{4}u_{2,x}r,\\
&\theta_{3,4}=-\frac{\alpha}{8}q^2r-\frac{\alpha}{8}u_2^2r+\frac{\alpha}{4}q_xs-\frac{\alpha}{4}rs-\frac{\alpha}{8}qu_1u_2-\frac{\alpha}{8}u_2^2r+\frac{\alpha}{8}q_{xxx}-\frac{\alpha}{8}r_{xx}+\frac{3\alpha}{16}q^2q_x\\
&\qquad\quad+\frac{\alpha}{16}(qu_2^2)_x+\frac{\alpha}{8}(qs)_x+\frac{u_2}{2}\int\Theta dx+\beta_2xr-\frac{\beta_2}{2}(xq)_x,\\
&\theta_{4,4}=-\frac{\alpha}{16}(q^2u_2)_x-\frac{\alpha}{16}(u_2^3)_x-\frac{\alpha}{8}(u_2s)_x-\frac{\alpha}{8}u_{2,xxx}-\frac{\alpha}{8}u_{1,xx}-\frac{\alpha}{4}q^2u_1-\frac{\alpha}{8}u_2^2u_1\\
&\qquad\quad-\frac{\alpha}{8}qru_2-\frac{\alpha}{4}u_{2,x}s-\frac{\alpha}{4}u_1s+\frac{q}{2}\int\Theta dx+\beta_2xu_1,\\
&\theta_{1,2\sigma+1}=\theta_{3,2\sigma+1}=\theta_{4,2\sigma+1}=0, \qquad \theta_{2,2\sigma}=\theta_{5,2\sigma}=\theta_{6,2\sigma}=0,\\
&\bar{\theta}_{2,1}=\frac{\bar{\alpha}}{2}q, \quad \bar{\theta}_{5,1}=\frac{\bar{\alpha}}{2}u_2, \quad \bar{\theta}_{3,1}=0, \quad \bar{\theta}_{1,1}=0, \quad \bar{\theta}_{2,2}=0,\\
&\bar{\theta}_{4,2}=\frac{\bar{\alpha}}{2}u_{2x}+\frac{\bar{\alpha}}{2}u_1, \quad \bar{\theta}_{3,2}=-\frac{\bar{\alpha}}{4}q_x+\frac{\bar{\alpha}r}{2}, \quad \bar{\theta}_{1,2}=-\frac{\bar{\alpha}}{4}q^2-\frac{\bar{\alpha}}{4}u_2^2+2\beta_2x,\\
&\bar{\theta}_{2,3}=-\frac{\bar{\alpha}}{4}q_{xx}+\frac{\bar{\alpha}}{4}r_x-\frac{\bar{\alpha}}{8}q^3-\frac{\bar{\alpha}}{8}qu_2^2-\frac{\bar{\alpha}}{4}qs+\beta_2xq,\\
&\bar{\theta}_{5,3}=-\frac{\bar{\alpha}}{8}q^2u_2-\frac{\bar{\alpha}}{8}u_2^3-\frac{\bar{\alpha}}{4}u_2s-\frac{\bar{\alpha}}{4}u_{2,xx}-\frac{\bar{\alpha}}{4}u_{1x}+\beta_2xu_2,
\end{aligned}$$

$$\begin{aligned}
&\bar{\theta}_{6,3}=-\frac{\bar{\alpha}}{4}qu_1-\frac{\bar{\alpha}}{4}u_2r+\int\Theta dx, \quad \Theta=\frac{\bar{\alpha}}{4}q_{xx}u_2-\frac{\bar{\alpha}}{4}u_{2,xx}q-\frac{\bar{\alpha}}{4}q_xu_1-\frac{\bar{\alpha}}{4}u_{2,x}r,\\
&\bar{\theta}_{3,4}=-\frac{\bar{\alpha}}{8}q^2r-\frac{\bar{\alpha}}{8}u_2^2r+\frac{\bar{\alpha}}{4}q_xs-\frac{\bar{\alpha}}{4}rs-\frac{\bar{\alpha}}{8}qu_1u_2-\frac{\bar{\alpha}}{8}u_2^2r+\frac{\bar{\alpha}}{8}q_{xxx}-\frac{\bar{\alpha}}{8}r_{xx}+\frac{3\bar{\alpha}}{16}q^2q_x\\
&\qquad +\frac{\bar{\alpha}}{16}(qu_2^2)_x+\frac{\bar{\alpha}}{8}(qs)_x+\frac{u_2}{2}\int\Theta dx+\beta_2xr-\frac{\beta_2}{2}(xq)_x,\\
&\bar{\theta}_{4,4}=-\frac{\bar{\alpha}}{16}(q^2u_2)_x-\frac{\bar{\alpha}}{16}(u_2^3)_x-\frac{\bar{\alpha}}{8}(u_2s)_x-\frac{\bar{\alpha}}{8}u_{2,xxx}-\frac{\bar{\alpha}}{8}u_{1,xx}-\frac{\bar{\alpha}}{4}q^2u_1-\frac{\bar{\alpha}}{8}u_2^2u_1-\frac{\bar{\alpha}}{8}qru_2\\
&\qquad -\frac{\bar{\alpha}}{4}u_{2,x}s-\frac{\bar{\alpha}}{4}u_1s+\frac{q}{2}\int\Theta dx+\beta_2xu_1+\frac{\beta_2}{2}(xu_2)_x,\\
&\bar{\theta}_{1,2\delta+1}=\bar{\theta}_{3,2\delta+1}=\bar{\theta}_{4,2\delta+1}=0, \quad \bar{\theta}_{2,2\delta}=\bar{\theta}_{5,2\delta}=\bar{\theta}_{6,2\delta}=0, \quad K_{2\delta+1}=0.
\end{aligned}$$

Note that

$$\begin{aligned}
\vec{V}_{1,+}^{(n)}=&\sum_{i=0}^{n}(\theta_{1,i}E_1(n-i)+\theta_{2,i}E_2(n-i)+\theta_{3,i}E_3(n-i)+\theta_{4,i}E_4(n-i)\\
&+\theta_{5,i}E_5(n-i)+\theta_{6,i}E_6(n-i))=\lambda^n\vec{V}_1-\vec{V}_{1,-}^{(n)},\\
\vec{V}_{2,+}^{(m)}=&\sum_{j=0}^{m}(\bar{\theta}_{1,j}E_1(m-j)+\bar{\theta}_{2,j}E_2(m-j)+\bar{\theta}_{3,j}E_3(m-j)+\bar{\theta}_{4,j}E_4(m-j)\\
&+\bar{\theta}_{5,j}E_5(m-j)+\bar{\theta}_{6,j}E_6(m-j))=\lambda^m\vec{V}_2-\vec{V}_{2,-}^{(m)}.
\end{aligned}$$

A direct calculation yields

$$\begin{aligned}
&-(\vec{V}_{1,+}^{(n)})_x+\vec{U}\times\vec{V}_{1,+}^{(n)}-(\vec{V}_{2,+}^{(m)})_x+\vec{U}\times\vec{V}_{2,+}^{(m)}+\frac{\partial\vec{U}}{\partial\lambda}\lambda_{t,+}^{(m)}\\
=&(-r\theta_{2,n}+u_1\theta_{5,n}-r\bar{\theta}_{2,m}+u_1\bar{\theta}_{5,m}-K_{m-1}s)E_1(-1)-K_msE_1(-2)\\
&+(r\theta_{1,n}-s\theta_{3,n}+r\bar{\theta}_{1,m}-s\bar{\theta}_{3,m})E_2(-1)\\
&+(s\theta_{2,n}-u_1\theta_{6,n}+s\bar{\theta}_{2,n}-u_1\bar{\theta}_{6,m})E_3(-1)-K_mrE_3(-2)-u_1K_mE_4(-2)\\
&+(r\theta_{6,n}-s\theta_{5,n}+r\bar{\theta}_{6,m}-s\bar{\theta}_{5,m}-K_{m-1}u_1)E_4(-1)\\
&+(-\theta_{1,n}u_1+s\theta_{4,n}-\bar{\theta}_{1,m}u_1+s\bar{\theta}_{4,m})E_5(-1)+(\theta_{3,n}u_1-r\theta_{4,n}+\bar{\theta}_{3,m}u_1-r\bar{\theta}_{4,m})E_6(-1)\\
&-(2\theta_{2,n+1}+2\bar{\theta}_{2,m+1})E_3(0)+(2\theta_{3,n+1}+2\bar{\theta}_{3,m+1})E_2(0)\\
&-(2\theta_{4,n+1}+2\bar{\theta}_{4,m+1})E_5(0)+(2\theta_{5,n+1}+2\bar{\theta}_{5,m+1})E_4(0). \qquad (3.10)
\end{aligned}$$

Choose $n=2\sigma+1$, $m=2\delta+1$, we find that the modified term Δ_n and Δ_m are taken as 0 so that $\vec{V}_{1,+}^{(n)}=\vec{V}_1^{(n)}$, $\vec{V}_{2,+}^{(m)}=\vec{V}_2^{(m)}$. Then

$$\begin{aligned}
&-(\vec{V}_1^{(n)})_x+\vec{U}\times\vec{V}_1^{(n)}-(\vec{V}_2^{(m)})_x+\vec{U}\times\vec{V}_2^{(m)}+\frac{\partial\vec{U}}{\partial\lambda}\lambda_{t,+}^{(m)}\\
=&(-r\theta_{2,2\sigma+1}+u_1\theta_{5,2\sigma+1}-r\bar{\theta}_{2,2\delta+1}+u_1\bar{\theta}_{5,\delta+1}-K_{2\delta}s)E_1(-1)
\end{aligned}$$

$$
\begin{aligned}
&+(s\theta_{2,2\sigma+1}-u_1\theta_{6,2\sigma+1}+s\bar{\theta}_{2,2\delta+1}-u_1\bar{\theta}_{6,2\delta+1})E_3(-1)\\
&+(r\theta_{6,2\sigma+1}-s\theta_{5,2\sigma+1}+r\bar{\theta}_{6,2\delta+1}-s\bar{\theta}_{5,2\delta+1}-K_{2\delta}u_1)E_4(-1)\\
&+(2\theta_{3,2\sigma+2}+2\bar{\theta}_{3,2\delta+2})E_2(0)-(2\theta_{4,2\sigma+2}+2\bar{\theta}_{4,2\delta+2})E_5(0).
\end{aligned}
\tag{3.11}
$$

Hence, the compatibility condition (3.4) of the spectral problem leads to the Lax integrable hierarchy

$$
\begin{pmatrix} q\\ r\\ s\\ u_1\\ u_2 \end{pmatrix}_{t_{\sigma,\delta}}
=\begin{pmatrix}
-(2\theta_{3,2\sigma+2}+2\bar{\theta}_{3,2\delta+2})\\
-(-r\theta_{2,2\sigma+1}+u_1\theta_{5,2\sigma+1}-r\bar{\theta}_{2,2\delta+1}+u_1\bar{\theta}_{5,\delta+1}-K_{2\delta}s)\\
-(s\theta_{2,2\sigma+1}-u_1\theta_{6,2\sigma+1}+s\bar{\theta}_{2,2\delta+1}-u_1\bar{\theta}_{6,2\delta+1})\\
-(r\theta_{6,2\sigma+1}-s\theta_{5,2\sigma+1}+r\bar{\theta}_{6,2\delta+1}-s\bar{\theta}_{5,2\delta+1}-K_{2\delta}u_1)\\
2\theta_{4,2\sigma+2}+2\bar{\theta}_{4,2\delta+2}
\end{pmatrix}.
\tag{3.12}
$$

When $\sigma=1$, $\delta=0$, (3.12) reduces to

$$
\begin{aligned}
q_{t,1,0}=&\frac{\alpha}{4}q^2r+\frac{\alpha}{4}u_2^2r-\frac{\alpha}{2}q_xs+\frac{\alpha}{2}rs+\frac{\alpha}{4}qu_1u_2+\frac{\alpha}{4}u_2^2r-\frac{\alpha}{4}q_{xxx}+\frac{\alpha}{4}r_{xx}+2\frac{\bar{\alpha}}{4}q_x\\
&-\frac{3\alpha}{8}q^2q_x-\frac{\alpha}{8}(qu_2^2)_x-\frac{\alpha}{4}(qs)_x-u_2\int\Theta dx-2\beta_2xr+\beta_2(xq)_x-2\frac{\bar{\alpha}r}{2},
\end{aligned}
\tag{3.13a}
$$

$$
\begin{aligned}
r_{t,1,0}=&-\frac{\alpha}{4}q_{xx}r+\frac{\alpha}{4}r_xr-\frac{\alpha}{8}q^3r-\frac{\alpha}{8}qru_2^2-\frac{\alpha}{4}qsr+\frac{\alpha}{8}q^2u_1u_2+\frac{\alpha}{8}u_2^3u_1\\
&+\frac{\alpha}{4}u_1u_2s+\frac{\alpha}{4}u_1u_{2,xx}+\frac{\alpha}{4}u_1u_{1x}+\frac{\bar{\alpha}}{2}qr-\frac{\bar{\alpha}}{2}u_1u_2,
\end{aligned}
\tag{3.13b}
$$

$$
\begin{aligned}
s_{t,1,0}=&\frac{\alpha}{4}q_{xx}s-\frac{\alpha}{4}r_xs+\frac{\alpha}{8}q^3s+\frac{\alpha}{8}qsu_2^2+\frac{\alpha}{4}qs^2-\frac{\bar{\alpha}}{4}qu_1^2-\frac{\bar{\alpha}}{4}u_1u_2r\\
&+u_1\int\Theta dx-\frac{\bar{\alpha}}{2}qs,
\end{aligned}
\tag{3.13c}
$$

$$
\begin{aligned}
u_{1,t,1,0}=&\frac{\alpha}{4}qru_1+\frac{\alpha}{4}u_2r^2-r\int\Theta dx+\frac{\bar{\alpha}}{2}u_2s+\beta_2u_1-\frac{\alpha}{8}q^2u_2s-\frac{\alpha}{8}u_2^3s\\
&-\frac{\alpha}{4}u_2s^2-\frac{\alpha}{4}u_{2,xx}s-\frac{\alpha}{4}u_{1x}s,
\end{aligned}
\tag{3.13d}
$$

$$
\begin{aligned}
u_{2,t,1,0}=&-\frac{\alpha}{8}(q^2u_2)_x-\frac{\alpha}{8}(u_2^3)_x-\frac{\alpha}{4}(u_2s)_x-\frac{\alpha}{4}u_{2,xxx}-\frac{\alpha}{4}u_{1,xx}-\frac{\alpha}{2}q^2u_1-\frac{\alpha}{4}u_2^2u_1-\frac{\alpha}{4}qru_2\\
&-\frac{\alpha}{2}u_{2,x}s-\frac{\alpha}{2}u_1s+q\int\Theta dx+\beta_2xu_1-\frac{\bar{\alpha}}{8}(q^2u_2)_x-\frac{\bar{\alpha}}{8}(u_2^3)_x-\frac{\bar{\alpha}}{4}(u_2s)_x\\
&-\frac{\bar{\alpha}}{4}u_{2,xxx}-\frac{\bar{\alpha}}{4}u_{1,xx}-\frac{\bar{\alpha}}{2}q^2u_1-\frac{\bar{\alpha}}{4}u_2^2u_1-\frac{\bar{\alpha}}{4}qru_2-\frac{\bar{\alpha}}{2}u_{2,x}s\\
&-\frac{\bar{\alpha}}{2}u_1s+q\int\Theta dx+2\beta_2xu_1+\beta_2(xu_2)_x.
\end{aligned}
\tag{3.13e}
$$

We still set $r=s=u_1=u_2=0$, (3.13) becomes again the same modified mKdV equation (2.9) except for their various coefficients. Similarly, when $\sigma=0$, $\delta=1$, (3.12) reduces again to the nonisospectral modified mKdV equation (2.54) except for their various coefficients.

The degree distributions for the second vector-product loop algebra are as follows

$$\begin{aligned} &E_1(n)=E_1\otimes\lambda^{2n}, && E_2(n)=E_2\otimes\lambda^{2n+1}, && E_3(n)=E_3\otimes\lambda^{2n+1},\\ &E_4(n)=E_4\otimes\lambda^{2n}, && E_5(n)=E_5\otimes\lambda^{2n}, && E_6(n)=E_6\otimes\lambda^{2n+1}. \end{aligned}$$

The resulting vector-product computations are presented in the following

$$\begin{aligned} &E_1(n)\times E_2(m)=E_3(m+n), && E_2(m)\times E_3(n)=E_1(m+n+1),\\ &E_3(m)\times E_1(n)=E_2(m+n), && E_1(m)\times E_4(n)=E_5(m+n),\\ &E_1(m)\times E_5(n)=-E_4(m+n), && E_1(m)\times E_6(n)=0,\\ &E_2(m)\times E_4(n)=0, && E_2(m)\times E_5(n)=E_6(m+n),\\ &E_2(m)\times E_6(n)=-E_5(m+n+1), && E_3(m)\times E_4(n)=-E_6(m+n),\\ &E_3(m)\times E_5(n)=0, && E_3(m)\times E_6(n)=E_4(m+n+1),\\ &E_4(m)\times E_5(n)=E_1(m+n), && E_4(m)\times E_6(n)=-E_3(m+n),\\ &E_4(m)\times E_6(n)=-E_3(m+n), && E_1(m)\times E_4(n)=E_5(m+n),\\ &E_5(m)\times E_6(n)=E_2(m+n). \end{aligned}$$

Via the second vector-product loop algebra, we consider the following spectral problem

$$\vec{U}=E_3(0)+qE_1(0)+rE_4(0)+uE_3(-1)+vE_6(-1), \tag{3.14a}$$

$$\begin{aligned}\vec{V}=&a_mE_1(-m)+b_mE_2(-m)+c_mE_3(-m)+d_mE_4(-m)\\ &+e_mE_5(-m)+f_mE_6(-m).\end{aligned} \tag{3.14b}$$

Then the stationary zero curvature equation (2.2) generates

$$a_{m,x}=-b_{m+1}+re_m-ub_m, \qquad b_{m,x}=-qc_m+a_m+ua_{m-1}-ve_{m-1}, \tag{3.15a}$$

$$c_{m,x}=qb_m-rf_m+vd_m, \qquad d_{m,x}=f_{m+1}-qe_m+uf_m-vc_m, \tag{3.15b}$$

$$e_{m,x}=qd_m-ra_m+vb_m, \qquad f_{m,x}=-d_m+rc_m-ud_{m-1}. \tag{3.15c}$$

Denote

$$\begin{aligned}\vec{V}_+^{(n)}=&\sum_{m=0}^{n}(a_mE_1(n-m)+b_mE_2(n-m)+c_mE_3(n-m)+d_mE_4(n-m)\\ &\quad+e_mE_5(n-m)+f_mE_6(n-m))\\ =&\lambda^{2n}\vec{V}-\vec{V}_-^{(n)}.\end{aligned}$$

A direct calculation reads

$$\begin{aligned}&-\vec{V}_{+,x}^{(n)}+\vec{U}\times\vec{V}_{+}^{(n)}\\ =&(ua_n-ve_n)E_2(-1)+vd_{n+1}E_3(-1)-ud_nE_6(-1)\\ &+b_{n+1}E_1(0)-f_{n+1}E_4(0).\end{aligned} \tag{3.16}$$

Taking

$$\vec{\Delta}_n=\frac{1}{q}(ua_n-ve_n)E_3(-1)+\frac{v}{qu}(ua_n-ve_n)E_6(-1),$$

so that $\vec{V}^{(n)}=\vec{V}_{+}^{(n)}+\vec{\Delta}_n$, a direct calculation gives

$$\begin{aligned}-\vec{V}_x^{(n)}+\vec{U}\times\vec{V}^{(n)}=&b_{n+1}E_1(0)-\left(f_{n+1}+\frac{v}{qu}(ua_n-ve_n)\right)E_4(0)\\ &-\left(\left(\frac{v(ua_n-ve_n)}{qu}\right)_x-\frac{r}{q}(ua_n-ve_n)+ud_n\right)E_6(-1)\\ &-\left(\left(\frac{ua_n-ve_n}{q}\right)_x+\frac{vr}{qu}(ua_n-ve_n)-vd_{n+1}\right)E_3(-1).\end{aligned} \tag{3.17}$$

Thus, the zero curvature equation (2.6) admits

$$\begin{pmatrix} q_t\\ r_t\\ u_t\\ v_t\end{pmatrix}=\begin{pmatrix} -b_{n+1}\\ f_{n+1}-\dfrac{v}{qu}(ua_n-ve_n)\\ \left(\dfrac{ua_n-ve_n}{q}\right)_x+\dfrac{vr}{qu}(ua_n-ve_n)-vd_{n+1}\\ \left(\dfrac{v(ua_n-ve_n)}{qu}\right)_x-\dfrac{r}{q}(ua_n-ve_n)+ud_n\end{pmatrix}. \tag{3.18}$$

Remark 3.1. Similar to the previous step, by taking initial values for (3.15), we can also concretely find the values of a_m, b_m, c_m, d_m, e_m and f_m so that the integrable hierarchy (3.18) can also be reduced to some concrete equations. Here we omit it.

4 Conclusions

Consideration of vector-product zero curvature equation instead of normal zero curvature equation led to some generalized integrable hierarchies. Starting with three-dimensional Lie algebra and its loop algebra, we derived the isospectral and nonisospectral mKdV integrable hierarchies which were reduced to the mKdV equation, coupled mKdV system and nonisospectral mKdV equation, respectively. It is significant to point that we made use of the variational method to create a new vector-product trace identity for which the Hamiltonian structure of the isospectral integrable hierarchy was worked out in Section 2.1.1. In Section 2.1.2, we constructed the Darboux transformation of the integrable system (2.8) which change the old solution into new solution. Note that the determining equations of nonclassical symmetries associated with the mKdV equation were introduced in Section 2.1.2. It follows that we emphasize that a nonclassical operator whose multiple is still a nonclassical operators after a complex calculation. Subsequently, we turned the problem of nonlocal symmetry of the original equation into the problem of local symmetry of corresponding covering equation so that the determining equations associated with nonlocal symmetry of nonisospectral mKdV equation can be singled out. In Section 3, a new vector-product Lie algebra of a reductive homogeneous space was constructed for which two corresponding loop algebra were followed to presented. It follows that two integrable hierarchies of evolution equations were obtained. Under the choice of specific parameters for equation hierarchy (3.12), we again derived the mKdV equation and nonisospectral mKdV equation which differ only in coefficients from those obtained in Section 2. In the future work, we plan to generalize the method introduced in this paper to generate high-dimensional integrable systems by following the ways in [37]. Then, the corresponding covering, nonlocal symmetry, and recursion operator of the reduced high-dimensional equation can all be worked out [38, 39].

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