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A C^1-conforming Gauss Collocation Method for Elliptic Equations and Superconvergence Analysis Over Rectangular Meshes

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Abstract. This paper is concerned with a C^1-conforming Gauss collocation approximation to the solution of a model two-dimensional elliptic boundary problem. Superconvergence phenomena for the numerical solution at mesh nodes, at roots of a special Jacobi polynomial, and at the Lobatto and Gauss lines are identified with rigorous mathematical proof, when tensor products of C^1 piecewise polynomials of degree not more than $k, k \geq 3$ are used. This method is shown to be superconvergent with $(2k-2)$-th order accuracy in both the function value and its gradient at mesh nodes, $(k+2)$-th order accuracy at all interior roots of a special Jacobi polynomial, $(k+1)$-th order accuracy in the gradient along the Lobatto lines, and k-th order accuracy in the second-order derivative along the Gauss lines. Numerical experiments are presented to indicate that all the superconvergence rates are sharp.

AMS subject classifications: 65N12, 65N15, 65N30

Key words: Hermite interpolation, C^1-conforming, superconvergence, Gauss collocation methods, Jacobi polynomials.

1 Introduction

The C^1-conforming Gauss collocation method, also known as orthogonal spline collocation (OSC) method or spline collocation at Gauss points, was first proposed and studied by de Boor and Swartz [21] for solving two-point boundary value problems. Since then, considerable advances have been made in the formulation, analysis and application of

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this method, especially OSC for partial differential equations such as elliptic equations, initial-boundary value problems for parabolic, hyperbolic and Schrödinger-type systems (see, e.g. [22,25,28–30]) and so on. Comparing with the counterpart C^1-conforming finite element method, the most attractive feature of the C^1 collocation method is the simple and fast calculation of the coefficients of the mass and stiffness matrices since no integrals need to be evaluated or approximated, as well as its desired superconvergence phenomena not shared by the C^1 finite element method. Compared to the C^0 type such as finite volume methods (FVMs) and finite element methods (FEMs) or L^2 type such as the discontinuous Galerkin (DG) method, the advantage of the C^1-conforming method lies in the continuity of the first-order derivative approximation across the element interface and the higher order approximation in the second-order derivative approximation, with the same or less degrees of freedom.

There are some theoretical a priori results for the C^1-conforming Gauss collocation method in the literature, we refer to [4,6,26,27] for an incomplete list of reference. In [4,27] the authors analyzed the C^1-conforming Gauss collocation method for two dimensional elliptic equations on the rectangular mesh and established existence, uniqueness of the numerical solution, and derived optimal error estimates in the H^2, H^1 and L^2-norms. Meanwhile, superconvergence property of the method has also been investigated. It was proved in [5] that the solution of the C^1-conforming Gauss collocation method for the two-point boundary value problem is superconvergent at nodes with an order of $\mathcal{O}(h^{2k-2})$. As for two dimensional elliptic problems, it was observed numerically in [7,8] that the gradient value at the mesh nodes on rectangles has the same convergence rate $\mathcal{O}(h^{2k-2})$. However, a theoretical proof of this remarkable property remains open. Only for a very special case, i.e. $k=3$ on uniform rectangular meshes, the authors in [3,5] proved a fourth-order accuracy for the gradient approximation at mesh nodes. Comparing with other numerical methods such as FEMs (see, e.g. [2,9,23,24,31,33]), FVMs (see, e.g. [10,14,16,19,34]), DG methods (see, e.g. [1,13,15,18,35]), spectral Galerkin methods (see, e.g. [36,37]) in the literature, the superconvergence study for the C^1-conforming Gauss collocation methods is far from satisfied and developed.

The main purpose of our current work is to present a full picture for superconvergence properties of the C^1 collocation method for second-order elliptic problems in the two-dimensional setting. We prove that the method achieves convergence rate $2k-2$ for both solution and its gradient at mesh nodes under quasi-uniform rectangular meshes for piecewise bi-k polynomial space. In other words, we extend the superconvergence results in [3,5] from a special case (i.e. $k=3$ on uniform rectangular meshes) to a more general case (i.e. any polynomial $k\geq 3$ on non-uniform rectangular meshes). In addition, some new superconvergence points and lines are discovered, which are identified as Lobatto and Gauss lines and roots of a generalized Jacobi polynomial. To be more precise, we prove that the method is superconvergent with order $k+2$ at roots of a generalized Jacobi polynomial for the solution approximation; with order $k+1$ at interior Lobatto lines for the gradient approximation; with order k at Gauss lines for the second-order derivative approximation. As a byproduct, a supercloseness result of the numerical solution towards a particular Jacobi projection of the exact solution is derived in all H^m-norms, $m=0,1,2$.

To establish the superconvergence results for the C^1-conforming Gauss collocation method, we first construct suitable basis functions (i.e. a special class of Jacobi polynomials) for the tensor product C^1 piecewise polynomial space. With the help of the special Jacobi polynomial, we then design a special Jacobi projection based on the truncated Jacobi expansion of the exact solution. Finally, we adopt the idea of correction function to prove that the numerical solution is superconvergent to this particularly designed Jacobi projection and thus shares the same superconvergent results with it. The key ingredient of our superconvergence analysis is the correction idea, which is motivated from its successful applications to FEMs, FVMs, and DG methods (see, e.g. [15–17]). However, due to the difference among these numerical schemes and different choice of approximation spaces, the correction function for the C^1 collocation method is very different from its C^0 counterpart methods in [15–17].

The rest of the paper is organized as follows. In Section 2, we present a C^1-conforming Gauss collocation method for two-dimensional elliptic equations over rectangular meshes. In Section 3, we prove the existence and uniqueness of the numerical scheme. In Section 4, we construct a C^1-conforming Jacobi projection of the exact solution and study the approximation and superconvergence properties of the special Jacobi projection. Section 5 is the main and most technical part, where optimal error estimates and superconvergence behavior at the mesh points (function and first-order derivative value approximations), at interior roots of Jacobi polynomials (function value approximation), at Lobatto lines (first-order derivative value approximation) and Gauss lines (the second-order derivative value approximation) are investigated. Numerical experiments supporting our theory are presented in Section 6. Some concluding remarks are provided in Section 7.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D\subset\Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D=\Omega$, we omit the index D; and if $p=2$, we set $W^{m,p}(D)=H^m(D)$, $\|\cdot\|_{m,p,D}=\|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D}=|\cdot|_{m,D}$. Notation $A\lesssim B$ implies that A can be bounded by B multiplied by a constant independent of the mesh size h.

2 A C^1-conforming Gauss collocation method

We consider the following convection-diffusion problem:

$$\begin{aligned} &\mathcal{L}u:=-\nabla\cdot(\alpha\nabla u)+\beta\cdot\nabla u+\gamma u=f && \text{in } \Omega=(a,b)\times(c,d),\\ &u=0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\alpha\geq\alpha_0>0, \gamma-\nabla\cdot\beta/2\geq 0, \gamma\geq 0, \beta=(\beta_1,\beta_2), \gamma\in L^\infty(\bar{\Omega})$, and f is a real-valued function defined on $\bar{\Omega}$. For simplicity, we assume that α,β,γ are all constants. The analysis can be generalized to the variable coefficient cases without any difficulty.

Let $a=x_0<x_1<\cdots<x_M=b$ and $c=y_0<y_1<\cdots<y_N=d$. For any positive integer r, we define $\mathbb{Z}_r=\{1,2,\ldots,r\}$, and denote by $\mathcal{T}_h$ the rectangular partition of Ω. That is

$$\mathcal{T}_h=\big\{\tau_{i,j}=[x_{i-1},x_i]\times[y_{j-1},y_j]:(i,j)\in\mathbb{Z}_M\times\mathbb{Z}_N\big\}.$$

For any $\tau\in\mathcal{T}_h$, we denote by h_τ^x,h_τ^y the lengths of x- and y-directional edges of τ, respectively. h is the maximal length of all edges, and $h_{\min}=\min_\tau(h_\tau^x,h_\tau^y)$. We assume that the mesh $\mathcal{T}_h$ is quasi-uniform in the sense that there exists a constant $c>0$ such that

$$h\le ch_{\min}.$$

We define the C^1 finite element space as follows:

$$V_h:=\big\{v\in C^1(\Omega):v|_\tau\in\mathbb{Q}_k(x,y)=\mathbb{P}_k(x)\times\mathbb{P}_k(y),\tau\in\mathcal{T}_h\big\},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree not more than k. Let

$$V_h^0:=\{v\in V_h:v|_{\partial\Omega}=0\}.$$

Define reference element $\hat{\tau}=[-1,1]\times[-1,1]$ and let $G_j,j\in\mathbb{Z}_{k-1}$ be Gauss points of degree $k-1$ (i.e. zeros of the Legendre polynomial L_{k-1}) in $[-1,1]$. Then $g_{i,j}^{\hat{\tau}}=(G_i,G_j)$, $i,j\in\mathbb{Z}_{k-1}$ constitute $(k-1)^2$ Gauss points in $\hat{\tau}$. Given $\tau\in\mathcal{T}_h$, let F_τ be the affine mapping from $\hat{\tau}$ to τ. Then Gauss points of degree $k-1$ in τ are

$$G_\tau=\big\{g_{i,j}^\tau:g_{i,j}^\tau=F_\tau(G_i,G_j),\ i,j\in\mathbb{Z}_{k-1}\big\}.$$

The Gauss-collocation method to (2.1) is: Find a $u_h\in V_h^0$ such that

$$\big(-\nabla\cdot(\alpha\nabla u_h)+\beta\cdot\nabla u_h+\gamma u_h\big)\big(g_{i,j}^\tau\big)=f\big(g_{i,j}^\tau\big),\quad (i,j)\in\mathbb{Z}_{k-1}\times\mathbb{Z}_{k-1},\quad \forall\tau\in\mathcal{T}_h. \tag{2.2}$$

To end with this section, we would like to relate the above equation (2.2) to its equivalent bilinear form, which serves as a basis in our later convergence analysis. For any $\tau\in\mathcal{T}_h$, denote by $w_{i,j}^\tau$ the associated Gauss weights corresponding to the Gauss points $g_{i,j}^\tau$, and $(\cdot,\cdot)_{*,\tau}$ the discrete inner product over τ, i.e.

$$(u,v)_{*,\tau}=\sum_{i,j=1}^{k-1}(uv)\big(g_{i,j}^\tau\big)w_{i,j}^\tau,\quad \forall u,v.$$

We define

$$(u,v)_*:=\sum_{\tau\in\mathcal{T}_h}(u,v)_{*,\tau}=\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k-1}(uv)\big(g_{i,j}^\tau\big)w_{i,j}^\tau,$$

and the bilinear form

$$a(u,v):=\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k-1}\big(-\nabla\cdot(\alpha\nabla u)+\beta\cdot\nabla u+\gamma u\big)\big(g_{i,j}^\tau\big)v\big(g_{i,j}^\tau\big)w_{i,j}^\tau=(\mathcal{L}u,v)_*. \tag{2.3}$$

For any $v\in\mathbb{Q}_{k-2}$, we multiply $v(g^\tau_{i,j})w^\tau_{i,j}$ in both side of (2.2) and sum up all elements to obtain

$$a(u_h,v)=(f,v)_*=\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k-1}f\big(g^\tau_{i,j}\big)v\big(g^\tau_{i,j}\big)w^\tau_{i,j},\quad \forall v\in\mathbb{Q}_{k-2}. \tag{2.4}$$

Similarly, if we choose $v\in\mathbb{Q}_{k-2}$ to be the associated Lagrange basis function corresponding to $g^\tau_{i,j}$ in (2.4), we get (2.2) immediately. In other words, the numerical scheme (2.2) is equivalent to (2.4).

3 Weak coercivity of the bilinear form

In this section, we study the property of the bilinear form $a(\cdot,\cdot)$, especially the weak coercivity of $a(\cdot,\cdot)$. Our later superconvergence analysis is based on this important property.

In each element $\tau\in\mathcal{T}_h$, we note that any function $v\in V_h$ is a polynomial and differentiable, and thus $\partial^i_x\partial^j_y v|\tau$ exists for all $i,j\le k$. Without causing confusion, we use the notation $\partial^i_x\partial^j_y v$ to represent the piecewise derivative function of v imposed on each $\tau\in\mathcal{T}_h$. For any function $v\in V^0_h$, we define

$$I(v):=(v,v_{xxyy})-(v,v_{xxyy})_*,\quad J(v):=(\triangle v,v_{xxyy})-(\triangle v,v_{xxyy})_*, \tag{3.1}$$

$$E(v):=(\beta\cdot\nabla v,v_{xxyy})_*-(\beta\cdot\nabla v,v_{xxyy}), \tag{3.2}$$

where $(u,v)=\sum_{\tau\in\mathcal{T}_h}\int_\tau(uv)(x,y)dxdy$ denotes the inner product of u,v.

Lemma 3.1. *For any $v\in V^0_h$ the following relations hold:*

$$I(v)\le 0,\quad J(v)\ge 0,\quad |E(v)|\le Ch\big(\|v_{xxy}\|^2_0+\|v_{yyx}\|^2_0\big).$$

Here C is a constant independent of the mesh size h.

The proof of the above lemma is given in the Appendix, see Section A.1.

Proposition 3.1. *The bilinear form $a(\cdot,\cdot)$ defined in* (2.3) *is weak coercive in the sense that*

$$|a(v,v_{xxyy})|\ge\frac{\alpha}{4}\big(\|v_{xyy}\|^2_0+\|v_{xxy}\|^2_0\big)-C\|v\|^2_1,\quad\forall v\in V^0_h, \tag{3.3}$$

where the constant C is independent of the mesh size h.

Proof. Recalling the definition of $a(\cdot,\cdot)$ and the conclusions in Lemma 3.1, we easily get

$$\begin{aligned}a(v,v_{xxyy})&\ge(-\alpha\triangle v+\beta\cdot\nabla v+\gamma v,v_{xxyy})+E(v)\\&\ge(\alpha-Ch)\big(\|v_{xyy}\|^2_0+\|v_{xxy}\|^2_0\big)+\gamma\|v_{xy}\|^2_0+(\beta\cdot\nabla v,v_{xxyy}),\end{aligned} \tag{3.4}$$

where $E(v)$ is defined in (3.2), and in the second step, we have used the integration by parts and the fact that

$$\partial_x^i v(x,c)=\partial_x^i v(x,d)=\partial_y^i v(a,y)=\partial_y^i v(b,y)=0, \quad \forall i\geq 0, \quad v\in V_h^0.$$

We next estimate the term $(\beta\cdot\nabla v, v_{xxyy})$. On the one hand, we have, from a direct calculation that

$$|(\beta\cdot\nabla v, v_{xxyy})|=|(v_{xy},\beta_1 v_{xxy}+\beta_2 v_{xyy})|\leq \frac{c_0}{\alpha}\|v_{xy}\|_0^2+\frac{\alpha}{4}\left(\|v_{xxy}\|_0^2+\|v_{xyy}\|_0^2\right)$$

with $c_0=\max(\beta_1^2,\beta_2^2)$. On the other hand, by the integration by parts, the inverse inequality, and the Cauchy-Schwarz inequality, we have

$$(v_{xy},v_{xy})=-(v_{xyy},v_x)\leq \epsilon\|v_{xyy}\|_0^2+\frac{1}{4\epsilon}\|v\|_1^2.$$

Here ϵ is a positive constant. By choosing a special ϵ satisfying $c_0\epsilon/\alpha\leq\alpha/4$, there exists a positive constant c_1 independent of h such that

$$|(\beta\cdot\nabla v, v_{xxyy})|\leq \frac{\alpha}{2}\left(\|v_{xxy}\|_0^2+\|v_{xyy}\|_0^2\right)+c_1\|v\|_1^2. \tag{3.5}$$

Plugging the above inequality into (3.4) leads to

$$\begin{aligned} a(v,v_{xxyy}) &\geq \left(\frac{\alpha}{2}-Ch\right)\left(\|v_{xyy}\|_0^2+\|v_{xxy}\|_0^2\right)+\gamma\|v_{xy}\|_0^2-c_1\|v\|_1^2 \\ &\geq \frac{\alpha}{4}\left(\|v_{xyy}\|_0^2+\|v_{xxy}\|_0^2\right)+\gamma\|v_{xy}\|_0^2-c_1\|v\|_1^2, \end{aligned}$$

provided that h is sufficiently small. This finishes our proof. □

Remark 3.1. By using the weak coercivity of the bilinear form and some Poincaré inequality, we can prove the uniqueness of the numerical solution of (2.2). Since the uniqueness of the numerical solution and the optimal error estimates have been studied in [4, 27], we omit the proof here and focus our attention on the superconvergence property of the numerical solution.

4 The truncated Jacobi projection

This section is dedicated to the introduction of a special C^1 truncated Jacobi projection of the exact solution u. The truncated Jacobi projection plays important role in our superconvergence analysis.

We begin with some orthogonal polynomials. Denote by L_n the Legendre polynomial of degree n on $[-1,1]$, and ϕ_{n+1} the Lobatto polynomial of degree $n+1$, which is defined by

$$\phi_{n+1}(s):=\int_{-1}^{s}L_n(s)ds=\frac{1}{2n+1}(L_{n+1}-L_{n-1})=\frac{1}{n(n+1)}(s^2-1)L_n'(s), \quad n\geq 1. \tag{4.1}$$

Define

$$J_{n+1}(s):=\int_{-1}^{s}\phi_n(s)ds=\frac{1}{2n-1}(\phi_{n+1}-\phi_{n-1})(s),\quad n\geq 3. \tag{4.2}$$

Actually, the original function $J_{n+1}(s)$ defined here is exactly the standard Jacobi polynomial $J_{n+1}^{-2,-2}(s)$ of degree $n+1$ (see, e.g. [32]), which is orthogonal with respect to the Jacobi weight function $\omega(s):=(1-s)^{-2}(1+s)^{-2}$.

Define the four Hermite interpolation basis functions on the interval $[-1,1]$ as follows:

$$J_0(s)=\frac{1}{4}(s+2)(1-s)^2,\quad J_1(s)=\frac{1}{4}(2-s)(1+s)^2,$$
$$J_2(s)=\frac{1}{4}(s+1)(1-s)^2,\quad J_3(s)=\frac{1}{4}(s-1)(1+s)^2.$$

Then $\{J_n\}_{n=0}^{\infty}$ constitutes the basis function of C^1 over $[-1,1]$. For any function $v\in C^1(\Omega)$, we suppose $v(x,y)$ has the following Jacobi expansion in each element $\tau_{ij},(i,j)\in\mathbb{Z}_M\times\mathbb{Z}_N$:

$$v(x,y)|_{\tau_{ij}}=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}v_{pq}J_{i,p}^{x}(x)J_{j,q}^{y}(y), \tag{4.3}$$

where v_{pq} are coefficients dependent on v, and

$$J_{i,p}^{x}(x)=J_p\left(\frac{2x-x_i-x_{i-1}}{h_i}\right)=J_p(s),\quad J_{j,p}^{y}(y)=J_p\left(\frac{2y-y_j-y_{j-1}}{h_j}\right)=J_p(s),\quad s\in[-1,1],$$

denote the Jacobi polynomial of degree p on $[x_{i-1},x_i]$ and $[y_{j-1},y_j]$, respectively. Now we define a truncated Jacobi projection $P_hv\in V_h$ of v as follows:

$$P_hv(x,y)|_{\tau_{i,j}}:=\sum_{p=0}^{k}\sum_{q=0}^{k}v_{pq}J_{i,p}^{x}(x)J_{j,q}^{y}(y). \tag{4.4}$$

Note that when $k=3$, the truncated Jacobi projection P_hv is exactly the Hermite interpolation of v.

Denote by $l_p,p\in\mathbb{Z}_k$ the Lobatto points of degree k in $[-1,1]$, (i.e. zeros of Lobatto polynomial ϕ_k). By the affine mapping F_τ from $\hat{\tau}$ to τ, the k^2 Lobatto points in τ are

$$l_\tau:=\{l_{i,j}^{\tau}:l_{i,j}^{\tau}=F_\tau(l_i,l_j),i,j\in\mathbb{Z}_k\}.$$

Then the Lobatto points on the whole domain Ω are defined as

$$\mathcal{L}:=\{z\in l_\tau,\tau\in\mathcal{T}_h\}.$$

Similarly, for $k\geq 3$, let $R_p,p\in\mathbb{Z}_{k-3}$ be the $k-3$ zeros of $J_{k+1}(s)$ except the point $s=-1,1$, and we define

$$R_\tau:=\{R_{i,j}^{\tau}:R_{i,j}^{\tau}=F_\tau(R_i,R_j),i,j\in\mathbb{Z}_{k-3}\},\quad \mathcal{R}:=\{z=R_\tau,\tau\in\mathcal{T}_h\}.$$

Denote by $\mathcal{E}_x^g, \mathcal{E}_y^g$ the Gauss line along the x-direction and the y-direction. That is,

$$\mathcal{E}_x^g := \{z = F_\tau(G_i, s) : s \in [-1,1], i \in \mathbb{Z}_{k-1}, \tau \in \mathcal{T}_h\},$$
$$\mathcal{E}_y^g := \{z = F_\tau(s, G_i) : s \in [-1,1], i \in \mathbb{Z}_{k-1}, \tau \in \mathcal{T}_h\}.$$

Similarly, the Lottabo line along the x-direction $\mathcal{E}_x^l$ and along the y-direction $\mathcal{E}_y^l$ on the whole domain Ω are defined as

$$\mathcal{E}_x^l := \{z = F_\tau(l_i, s) : s \in [-1,1], i \in \mathbb{Z}_k, \tau \in \mathcal{T}_h\},$$
$$\mathcal{E}_y^l := \{z = F_\tau(s, l_i) : s \in [-1,1], i \in \mathbb{Z}_k, \tau \in \mathcal{T}_h\}.$$

We have the following approximation properties for the Jacobi projection (see [12]).

Proposition 4.1. *For any function $v \in W^{l,\infty}(\Omega) \cap C^1$, assume that $P_h v$ is the truncated Jacobi projection of v defined by* (4.4). *Then for any $r \le \min(k+1,l)$,*

$$\begin{aligned}
&\|v - P_h v\|_{m,p} \lesssim h^m \|u\|_{m,p}, && m \le \min(k+1,l), \quad p = 2,\infty, \\
&(v - P_h v)(x_i, y_j) = 0, && |(v - P_h v)(z_0)| \lesssim h^r \|u\|_{r,\infty}, \\
&\nabla(v - P_h v)(x_i, y_j) = 0, && |\partial_x (v - P_h v)(z_1)| + |\partial_y (v - P_h v)(z_2)| \lesssim h^{r-1} \|u\|_{r,\infty}, \\
&\left|\partial_{xx}^2 (v - P_h v)(z_3)\right| + \left|\partial_{yy}^2 (v - P_h v)(z_4)\right| + \left|\partial_{xy}^2 (v - P_h v)(z_5)\right| \lesssim h^{r-2} \|u\|_{r,\infty},
\end{aligned} \tag{4.5}$$

where $z_0 \in \mathcal{R}, z_1 \in \mathcal{E}_x^l, z_2 \in \mathcal{E}_y^l, z_3 \in \mathcal{E}_x^g, z_4 \in \mathcal{E}_y^g, z_5 \in \mathcal{L}$.

5 Superconvergence analysis

In this section, we study the superconvergence properties of the C^1 Gauss collocation method. Our analysis is along this line: We first prove that the numerical solution is super-close to a special projection u_I of the exact solution in the H^2-norm, and thus shares the same superconvergence properties of u_I; then we use the approximation properties of u_I and the supercloseness results between u_h and u_I to establish the superconvergence results of the numerical solution u_h.

5.1 Construction of a special projection u_I

Define

$$W_h := \{v \in L^2(\Omega) : v|_\tau \in \mathbb{Q}_{k-2}(x,y) = \mathbb{P}_{k-2}(x) \times \mathbb{P}_{k-2}(y), \tau \in \mathcal{T}_h\}. \tag{5.1}$$

To construct the special projection $u_I \in V_h$ superclose to u_h in the H^2-norm, we notice that $\partial_x^2 \partial_y^2 (u_I - u_h) \in W_h$, and then use the homogenous boundary condition, (3.3) and the orthogonality $a(u - u_h, \theta) = 0$ for any $\theta \in W_h$ to get

$$\begin{aligned}
|u_h - u_I|_2^2 &\lesssim \|\triangle u_I - \triangle u_h\|_0^2 \\
&\lesssim a\big(u_I - u_h, \partial_x^2 \partial_y^2 (u_I - u_h)\big) + \|u_I - u_h\|_1 \\
&= a\big(u_I - u, \partial_x^2 \partial_y^2 (u_I - u_h)\big) + \|u_I - u_h\|_1.
\end{aligned}$$

In other words, to achieve our superconvergence goal, we need to construct a u_I such that the right-hand side term $a(u-u_I,\theta),\theta\in W_h$ is of high order.

Define

$$u_I = P_h u - w_h, \tag{5.2}$$

where $w_h\in V_h^0$ is a function to be determined. Note that

$$a(u-u_I,\theta)=a(u-P_hu,\theta)+a(w_h,\theta).$$

Consequently, our ultimate goal is to design a special function w_h to correct the error bound of $a(u-P_hu,\theta)$. We also call w_h the correction function.

The following theorem indicates the existence of the correction function w_h.

Theorem 5.1. *Let $u\in W^{2k+1,\infty}(\Omega)$ is the solution of* (2.1). *There exists a $w_h\in V_h^0$ such that*

$$\|w_h\|_{0,\infty}\lesssim h^{\min(k+2,2k-2)}\|u\|_{2k+1,\infty}, \quad \|w_h\|_{1,\infty}+h\|w_h\|_{2,\infty}\lesssim h^{k+1}\|u\|_{2k+1,\infty}, \tag{5.3}$$

$$|w_h(x_i,y_j)|+|\nabla w_h(x_i,y_j)|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}. \tag{5.4}$$

Furthermore, for any $\theta\in W_h$ there holds

$$|a(u-u_I,\theta)|=|a(u-P_hu+w_h,\theta)|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta\|_0. \tag{5.5}$$

The proof of Theorem 5.1 is given in the next subsection.

5.2 Superconvergence results

Thanks to the constructions of the correction function w_h and u_I, we are ready to present the superconvergence results for the numerical solution u_h.

Define

$$\xi := u_I - u_h. \tag{5.6}$$

Lemma 5.1. *Assume that $u\in W^{2k+1,\infty}(\Omega)$ is the solution of* (2.1). *Then*

$$\|\nabla\xi\|_0\lesssim h^{2k-2}\|u\|_{2k+1,\infty}+h(\|\xi_{xxy}\|_0+\|\xi_{yyx}\|_0). \tag{5.7}$$

Proof. First, we consider the following dual problem: Given any $\zeta\in[C^1(\Omega)]^2$, let ψ be the solution of the following dual problem:

$$\begin{aligned} &-\nabla\cdot(\alpha\nabla\psi)-\beta\cdot\nabla\psi+\gamma\psi=-\nabla\cdot\zeta && \text{in } \Omega,\\ &\psi=0 && \text{on } \partial\Omega. \end{aligned} \tag{5.8}$$

Using the integration by parts, for any $v\in V_h^0$, we have

$$\begin{aligned}(\nabla v,\zeta)&=-(v,\nabla\cdot\zeta)=\big(v,-\nabla\cdot(\alpha\nabla\psi)-\beta\cdot\nabla\psi+\gamma\psi\big)\\ &=\big(-\nabla\cdot(\alpha\nabla v)+\beta\cdot\nabla v+\gamma v,\psi-\bar\psi+\bar\psi\big),\end{aligned}$$

where $\bar{\psi} \in \mathbb{Q}_0$ denotes the cell average of ψ, i.e.

$$\bar{\psi}|_\tau = \frac{1}{|\tau|}\int_\tau \psi dxdy.$$

Since $(-\nabla\cdot(\alpha\nabla v)+\beta\cdot\nabla v+\gamma v)\bar{\psi} \in \mathbb{Q}_k$ and the $(k-1)$-point Gauss numerical quadrature is exact for polynomial of degree $2k-3$, then

$$\big(-\nabla\cdot(\alpha\nabla v)+\beta\cdot\nabla v+\gamma v,\bar{\psi}\big) = \big(-\nabla\cdot(\alpha\nabla v)+\beta\cdot\nabla v+\gamma v,\bar{\psi}\big)_* = a(v,\bar{\psi}), \tag{5.9}$$

and thus

$$\begin{aligned} |(\nabla v,\zeta)| &\lesssim h(\|v_{xx}\|_0+\|v_{yy}\|_0+\|v\|_1)\|\psi\|_1+|a(v,\bar{\psi})| \\ &\lesssim h(\|v_{xx}\|_0+\|v_{yy}\|_0+\|v\|_1)\|\psi\|_1+|a(v,\bar{\psi})|, \quad \forall v\in V_h^0. \end{aligned} \tag{5.10}$$

Now we choose $v=\xi\in V_h^0$ in the above inequality and use (5.5) to obtain

$$\begin{aligned} |(\nabla\xi,\zeta)| &\lesssim h(\|\xi_{xx}\|_0+\|\xi_{yy}\|_0+\|\xi\|_1)\|\psi\|_1+|a(u-u_I,\bar{\psi})| \\ &\lesssim h(\|\xi_{xx}\|_0+\|\xi_{yy}\|_0+\|\xi\|_1)\|\zeta\|_0+h^{2k-2}\|u\|_{2k+1,\infty}\|\zeta\|_0, \end{aligned}$$

where in the last step, we have used regularity result $\|\psi\|_1 \lesssim \|\nabla\cdot\zeta\|_{-1} \lesssim \|\zeta\|_0$. Since the set of all such ζ is dense in $L^2(\Omega)$, the above inequality indicates that

$$\|\nabla\xi\|_0 \lesssim h(\|\xi_{xx}\|_0+\|\xi_{yy}\|_0)+h^{2k-2}\|u\|_{2k+1,\infty} \tag{5.11}$$

for sufficiently small h. On the other hand, note that for any function $v\in V_h^0, \partial_x^i v, i\geq 1$ is continuous about y satisfying $\partial_x^i v(x,c)=\partial_x^i v(x,d)=0$. Similarly, $\partial_y^i v, i\geq 1$ is a continuous function about x satisfying $\partial_y^i v(a,y)=\partial_y^i v(b,y)$. Then

$$v_{xx}(x,y)=\int_c^y v_{xxy}(x,y)dy, \quad v_{yy}(x,y)=\int_a^x v_{yyx}(x,y)dx.$$

By the Poincaré inequality,

$$\|v_{xx}\|_0+\|v_{yy}\|_0 \lesssim \|v_{xxy}\|_0+\|v_{yyx}\|_0. \tag{5.12}$$

Then the desired result (5.7) follows by substituting (5.12) into (5.11). $\square$

Theorem 5.2. *Assume that $u\in W^{2k+1,\infty}(\Omega)$ is the solution of* (2.1), *and u_h is the solution of* (2.2). *The following superconvergence properties hold true.*

1. *Supercloseness results between u_h and the special truncated Jacobi projection of $P_h u$ in all H^2, H^1, L^2-norms*

$$\begin{aligned} &\|u_h-P_hu\|_1+h\|u_h-P_hu\|_2 \lesssim h^{k+1}\|u\|_{2k+1,\infty}, \\ &\|u_h-P_hu\|_0 \lesssim h^{\min(k+2,2k-2)}\|u\|_{2k+1,\infty}. \end{aligned} \tag{5.13}$$

2. *Superconvergence of the function value and the first-order derivative at nodes, i.e.*

$$e_{u,n}+e_{\nabla u,n}\lesssim h^{2k-2}\|u\|_{2k+1,\infty}, \tag{5.14}$$

where

$$e_{v,n}=\left(\frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}(v-v_h)^2(x_i,y_j)\right)^{\frac{1}{2}},\quad v=u,\nabla u.$$

3. *Superconvergence of function value approximation on roots of $J_{k+1}^{-2,-2}(x)J_{k+1}^{-2,-2}(y)$, i.e.*

$$e_{u,J}:=\left(\frac{1}{NM}\sum_{z\in\mathcal{R}}(u-u_h)^2(z)\right)^{\frac{1}{2}}\lesssim h^{\min(k+2,2k-2)}\|u\|_{2k+1,\infty}. \tag{5.15}$$

4. *Superconvergence of first and second derivative value approximations on Lobatto and Gauss line, respectively. That is,*

$$e_{\nabla u,l}\lesssim h^{k+1}\|u\|_{2k+1,\infty},\quad e_{\triangle u,g}\lesssim h^{k}\|u\|_{2k+1,\infty}, \tag{5.16}$$

where

$$e_{\nabla u,l}=\left(\frac{1}{N_x}\sum_{z_0\in\mathcal{E}_x^l}\partial_x(u-u_h)^2(z_0)+\frac{1}{N_y}\sum_{z_1\in\mathcal{E}_y^l}\partial_y(u-u_h)^2(z_1)\right)^{\frac{1}{2}},$$

$$e_{\triangle u,g}=\left(\frac{1}{M_x}\sum_{z_0\in\mathcal{E}_x^g}\partial_{xx}^2(u-u_h)^2(z_0)+\frac{1}{M_y}\sum_{z_1\in\mathcal{E}_y^g}\partial_{yy}^2(u-u_h)^2(z_1)\right)^{\frac{1}{2}}.$$

Here N_x,N_y,M_x,M_y denote the cardinalities of $\mathcal{E}_x^l,\mathcal{E}_y^l,\mathcal{E}_x^g,\mathcal{E}_x^g$, respectively.

Proof. First, by choosing $v=\xi$ in (3.3) and using the orthogonality $a(u-u_h,\theta)=0,\theta\in W_h$ and (5.5), we have

$$\begin{aligned}\|\xi_{xxy}\|_0^2+\|\xi_{xyy}\|_0^2&\lesssim\|\xi\|_1^2+a(u-u_I,\xi_{xxyy})\\&\lesssim\|\xi\|_1^2+h^{2k-3}\|u\|_{2k+1,\infty}\|\xi_{xxy}\|_0,\end{aligned} \tag{5.17}$$

where in the last step, we have used the inverse inequality $\|\xi_{xxyy}\|_0\lesssim h^{-1}\|\xi_{xyy}\|_0$. Substituting (5.7) into (5.17), we have

$$\|\xi_{xxy}\|_0+\|\xi_{xyy}\|_0\lesssim h^{2k-3}\|u\|_{2k+1,\infty},$$

which yields, together with (5.7) and (5.12) that

$$\begin{aligned}&\|\nabla\xi\|_0\lesssim h^{2k-2}\|u\|_{2k+1,\infty},\\&\|\xi\|_2\lesssim\|\xi_{xx}\|_0+\|\xi_{yy}\|_0\lesssim h^{2k-3}\|u\|_{2k+1,\infty}.\end{aligned}$$

Since $\xi=0$ on $\partial\Omega$, we have from the Poincaré inequality,

$$\|\xi\|_0 \lesssim \|\xi\|_1 \lesssim h^{2k-2}\|u\|_{2k+1,\infty}.$$

Then (5.13) follows from (5.3) and the triangle inequality.

By (4.5), (5.4) and the inverse inequality, we get

$$\begin{aligned}|(u-u_h)(x_i,y_j)| &= |(P_hu+w_h-u_h)(x_i,y_j)-w_h(x_i,y_j)|\\ &\lesssim h^{-1}\|\xi\|_{0,\tau_{i,j}}+h^{2k-2}\|u\|_{2k+1,\infty},\end{aligned}$$

and thus

$$e_{u,n} \lesssim \|\xi\|_0+h^{2k-2}\|u\|_{2k+1,\infty} \lesssim h^{2k-2}\|u\|_{2k+1,\infty}.$$

Following the same argument, we have from (4.5) and (5.3)-(5.4) that

$$\begin{aligned}&e_{\nabla u,n} \lesssim \|\nabla\xi\|_0+h^{2k-2}\|u\|_{2k+1,\infty},\\ &e_{u,J} \lesssim \|\xi\|_0+h^{\min(k+2,2k-2)}\|u\|_{2k+1,\infty},\\ &e_{\nabla u,l} \lesssim \|\nabla\xi\|_0+\|w_h\|_{1,\infty}+h^{k+1}\|u\|_{k+2,\infty},\\ &e_{\triangle u,g} \lesssim \|\xi\|_2+\|w_h\|_{2,\infty}+h^{k}\|u\|_{k+2,\infty}.\end{aligned}$$

Then (5.14)-(5.16) follow from the estimates of $\|\xi\|_m, m\le 2$. This finishes our proof. $\square$

For $k\ge 4$, we have the following point-wise superconvergent error estimates.

Corollary 5.1. *Assume that $u\in W^{2k+1,\infty}(\Omega)$ is the solution of* (2.1), *and u_h is the solution of* (2.2). *Then for sufficiently small h,*

$$\begin{aligned}&|(u-u_h)(z)| \lesssim h^{k+2}\max\left(1,h^{k-4}\ln h^{\frac{1}{2}}\right)\|u\|_{2k+1,\infty},\\ &|\partial_x(u-u_h)(z_0)|+|\partial_y(u-u_h)(z_1)| \lesssim h^{k+1}\|u\|_{2k+1,\infty},\\ &\left|\partial^2_{xx}(u-u_h)(z_2)\right|+\left|\partial^2_{yy}(u-u_h)(z_3)\right|+\left|\partial^2_{xy}(u-u_h)(z_4)\right| \lesssim h^k\|u\|_{2k+1,\infty}\end{aligned}$$

with $z\in\mathcal{R}, z_0\in\mathcal{E}^l_x, z_1\in\mathcal{E}^l_y, z_2\in\mathcal{E}^g_x, z_3\in\mathcal{E}^g_y, z_4\in\mathcal{L}$.

Here we omit the proof and refer to [12] for the same argument.

5.3 Proof of Theorem 5.1

To construct the correction function $w_h\in V_h^0$ satisfying the conclusion of Theorem 5.1, we first note that, from (4.3)-(4.4),

$$(u-P_hu)(x,y)|_{\tau_{ij}} = \sum_{p=k+1}^{\infty}\sum_{q=k+1}^{\infty} u_{pq}J^x_{i,p}(x)J^y_{j,q}(y) = (E^xu+E^yu-E^xE^yu)(x,y), \tag{5.18}$$

where

$$\begin{aligned}
E^x u(x,y)|_{\tau_{ij}} &= \sum_{p=k+1}^{\infty}\sum_{q=0}^{\infty} u_{pq} J^x_{i,p}(x) J^y_{j,q}(y),\\
E^y u(x,y)|_{\tau_{ij}} &= \sum_{p=0}^{\infty}\sum_{q=k+1}^{\infty} u_{pq} J^x_{i,p}(x) J^y_{j,q}(y),\\
E^y E^x u(x,y)|_{\tau_{ij}} &= \sum_{p=k+1}^{\infty}\sum_{q=k+1}^{\infty} u_{pq} J^x_{i,p}(x) J^y_{j,q}(y).
\end{aligned}$$

It has been proved in [12] that the term E^yE^xu is of high-order, i.e.

$$\|E^xE^yu\|_{0,m}+h\|E^xE^yu\|_{1,m}+h^2\|E^xE^yu\|_{2,m}\lesssim h^{k+1+l}\|u\|_{k+1+l,m}, \quad 0\le l\le k+1. \tag{5.19}$$

Then

$$a(u-P_hu+w_h,\theta)=a(E^xu,\theta)+a(E^yu,\theta)+a(w_h,\theta)+\mathcal{O}(h^{k+1+l})\|\theta\|_0.$$

In other words, the key ingredient in the proof of Theorem 5.1 is to design a correction function w_h to improve the error bound $a(E^xu,\theta)+a(E^yu,\theta)$. In the rest of this subsection, we separately construct a function w_h^x and w_h^y such that $w_h=w_h^x+w_h^y$ and both the errors $a(E^xu+w_h^x,\theta)$ and $a(E^yu+w_h^y,\theta)$ are of high-order.

5.3.1 Construction of the function w_h^x for $a(E^xu,\theta)$

We begin with some preliminaries. First, let

$$\mathcal{L}_1:=-\alpha\partial_{yy}+\beta\cdot\nabla+\gamma, \quad \mathcal{L}_2:=-\alpha\partial_{yy}+\beta_2\partial_y+\gamma. \tag{5.20}$$

Second, we denote by

$$\begin{aligned}
B_i^x &= [x_{i-1},x_i]\times[c,d], \quad i\in\mathbb{Z}_M,\\
B_j^y &= [a,b]\times[x_{j-1},x_j], \quad j\in\mathbb{Z}_N
\end{aligned}$$

the element band along the x-direction and y-direction, respectively. Denoting $h_i^x=x_i-x_{i-1}, h_j^y=y_j-y_{j-1}$ and $(u,v)_{*,\tau_i^x}$ and $(u,v)_{*,\tau_j^y}$ the $(k+1)$-point Gauss numerical quadrature on τ_i^x and τ_j^y, respectively. That is,

$$(u,v)_{*,\tau_i^x}:=\sum_{m=1}^{k-1}(uv)(G^x_{i,m},y)w^x_{i,m}, \quad (u,v)_{*,\tau_j^y}:=\sum_{m=1}^{k-1}(uv)(x,G^y_{j,m})w^y_{j,m},$$

where $G^x_{i,m}$ and $w^x_{i,m}$ denote the Gauss points and corresponding Gauss weights over τ_i^x. Similarly for $G^y_{j,m}$ and $w^y_{j,m}$.

For any $p\ge k$, we denote by $\mathcal{I}_p^xv\in\mathbb{P}_p\cap C^0$ the interpolation function of v satisfying

$$\mathcal{I}_p^xv|_{\tau_i^x}\left(G^x_{i,m},y\right)=v\left(G^x_{i,m},y\right), \quad m\in\mathbb{Z}_{k-1}, \quad y\in[c,d]. \tag{5.21}$$

Third, we define

$$W_h^y := \left\{ v(y) \in C^1([c,d]) : v|_{\tau_j^y} \in \mathbb{P}_k(y), v(c) = v(d) = 0, j \in \mathbb{Z}_N \right\}. \tag{5.22}$$

Lemma 5.2 ([11]). *Given any smooth function g, assume that $v(y) \in W_h^y$ is the solution of the following problem:*

$$-\sum_{j=1}^{N} (v,\theta)_{*,\tau_j^y} = \sum_{j=1}^{N} (\zeta,\theta)_{*,\tau_j^y}, \quad \forall \theta \in \mathbb{P}_{k-2}(\tau_j^y). \tag{5.23}$$

Then $v(y)$ is well defined. Moreover, there holds

$$\|\partial_y^n v\|_{0,\infty,[c,d]} \lesssim \|\partial_y^n \zeta\|_{0,\infty,[c,d]}, \quad \forall n \leq k. \tag{5.24}$$

Now we are ready to construct the correction function w_h^x. Given any l, where $1 \leq l \leq k-2$, we define a sequence of function $w_l \in V_h$ for $1 \leq l \leq k-2$ as follows:

$$\partial_{xx} w_l|_{B_i^x} = \sum_{p=1}^{k-2} c_{i,p}^l(y) L_{i,p}(x), \quad \partial_x w_l(x_i,y) = 0, \quad i \in \mathbb{Z}_M, \quad w_l(a,y) = 0, \quad \forall y \in [c,d], \tag{5.25}$$

where $L_{i,p}(x)$ denotes the Legendre polynomial of degree p on τ_i^x, and $c_{i,p}^l(y) \in \mathbb{P}_k(y)$ is the solution of (5.23) with the right-hand function

$$\zeta = \zeta_{i,p}^l(y) := \begin{cases} \dfrac{2p+1}{\alpha h_i^x} \left(\mathcal{L} E^x u, L_{i,p} \right)_{*,\tau_i^x}, & \text{if} \quad l = 1, \\ \dfrac{2p+1}{\alpha h_i^x} \left(\mathcal{L}_1 w_{l-1}, L_{i,p} \right)_{*,\tau_i^x}, & \text{if} \quad 1 < l \leq k-2. \end{cases} \tag{5.26}$$

Using (5.25) and properties of Legendre and Lobatto polynomials in (4.1)-(4.2), we have

$$\partial_x w_l|_{B_i^x} = \int_{x_{i-1}}^{x} \partial_{xx} w_l dx = \frac{h_i^x}{2} \sum_{p=2}^{k-1} c_{i,p-1}^l(y) \phi_{i,p}(x), \tag{5.27}$$

$$w_l|_{B_i^x} = \int_a^x \partial_x w_l dx = \frac{h_i^x}{2} c_{i,1}^l(y) \int_a^x \phi_{i,2}(x) dx + \left(\frac{h_i^x}{2} \right)^2 \sum_{p=3}^{k-1} c_{i,p-1}^l(y) J_{i,p+1}^x(x). \tag{5.28}$$

Here $\phi_{i,p}, J_{i,p}^x$ separately denotes the Lobatto and Jacobi polynomials of degree p on τ_i^x.

We have the following properties for the specially defined functions $w_l, 1 \leq l \leq k-2$.

Lemma 5.3. *Let $w_l, 1 \leq l \leq k-2$, be the sequence of functions defined by* (5.25)-(5.28). *If $u \in W^{2k+1,\infty}(\Omega)$, then*

$$\left\| \partial_y^n c_{i,p}^l \right\|_{0,\infty} \lesssim h^m \left\| \partial_y^n u \right\|_{m+2,\infty}, \quad m \leq \mu_{l,p} = \max(2k-2-p, k+l-1), \quad \forall n \tag{5.29}$$

with $c_{i,p}^l$ the same as that in (5.25). *Consequently,*

$$|w_l(x_i,y_j)| + |\nabla w_l(x_i,y_j)| \lesssim h^{2k-2} \|u\|_{2k+1,\infty}, \tag{5.30}$$

$$\|\nabla w_l\|_{0,\infty} \lesssim h^{k+l}\|u\|_{k+l+1,\infty}, \quad \left\|\partial_y^n w_l\right\|_{0,\infty} \lesssim h^{m'}\left\|\partial_y^n u\right\|_{m',\infty} \tag{5.31}$$

with $m' \leq \min(k+l+1, 2k-2)$.

The proof of Lemma 5.3 is given in the Appendix, see Section A.2.
Define

$$w_h^x(x,y) := \sum_{l=1}^{k-2} w_l(x,y). \tag{5.32}$$

Proposition 5.1. *Let* $u \in W^{2k+1,\infty}$ *be the solution of* (2.1)*, and* $w_h^x \in V_h$ *be defined in* (5.32), (5.28)*. Then*

$$\left|a(E^x u + w_h^x, \theta)\right| \lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta\|_0, \quad \forall \theta \in W_h. \tag{5.33}$$

Proof. First, note that any function $\theta \in W_h$ can be decomposed into two terms, i.e. $\theta = \theta_0 + \theta_1$ with

$$\theta_1|_\tau \in \left(\mathbb{P}_{k-2}(x) \setminus \mathbb{P}_0(x)\right) \times \mathbb{P}_{k-2}(y), \quad \theta_0|_\tau \in \mathbb{P}_0(x) \times \mathbb{P}_{k-2}(y).$$

By letting $\theta_1 = L_{i,q}(x)v(y), q = 1,2,\ldots,k-2, v_p(y) \in \mathbb{P}_{k-2}(y)$ and using (5.23), (5.26) and the fact that the $k-1$ point Gauss numerical quadrature is exact for polynomials of degree $2k-3$, we get for all $1 \leq l \leq k-2$ that

$$\begin{aligned}
\alpha(\partial_{xx} w_l, \theta_1)_{*,\tau_{i,j}} &= \alpha \sum_{p=1}^{k-2} \left(c_{i,p}^l(y), v(y)\right)_{*,\tau_j^y} (L_{i,p}, L_{j,q})_{*,\tau_i^x} = \frac{\alpha h_i^x}{2q+1}\left(c_{i,q}^l(y), v(y)\right)_{*,\tau_j^y} \\
&= \frac{\alpha h_i^x}{2q+1}\left(\zeta_{i,q}^l, v(y)\right)_{*,\tau_{i,j}} = \begin{cases} \left(\mathcal{L}E^x u, \theta_1\right)_{*,\tau_{i,j}}, & \text{if} \quad l = 1, \\ (\mathcal{L}_1 w_{l-1}, \theta_1)_{*,\tau_{i,j}}, & \text{if} \quad 1 < l \leq k-2. \end{cases}
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left|a\left(E^x u + w_h^x, \theta_1\right)\right| &= \left|\left(\mathcal{L}E^x u, \theta_1\right)_* + \sum_{l=1}^{k-2}(-\alpha\partial_{xx} w_l, \theta_1)_* + (\mathcal{L}_1 w_l, \theta_1)_*\right| \\
&= |(\mathcal{L}_1 w_{k-2}, \theta_1)_*| \lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta_1\|_0.
\end{aligned}$$

Here in the last step, we have used (5.31).

On the other hand, for all $\theta_0 \in \mathbb{P}_0(x) \times \mathbb{P}_{k-2}(y)$, we use the property of Gauss numerical quadrature to obtain that

$$\begin{aligned}
a\left(E^x u + w_h^x, \theta_0\right) &= \left(\mathcal{L}E^x u, \theta_0\right)_* + \sum_{l=1}^{k-2}(\mathcal{L}w_l, \theta_0)_* \\
&= \left(\mathcal{I}_{2k-3}^y \mathcal{I}_{2k-3}^x \mathcal{L}E^x u, \theta_0\right) + \sum_{l=1}^{k-2}\left(\mathcal{I}_{2k-3}^y \mathcal{L}w_l, \theta_0\right) = \sum_{i=1}^{4} I_i,
\end{aligned}$$

where

$$I_1=\left(\mathcal{I}^y_{2k-3}\mathcal{I}^x_{2k-3}\mathcal{L}E^xu-\mathcal{L}E^xu,\theta_0\right),\quad I_2=\left(\mathcal{L}E^xu,\theta_0\right),$$
$$I_3=\sum_{l=1}^{k-2}\left(\mathcal{I}^y_{2k-3}\mathcal{L}w_l-\mathcal{L}w_l,\theta_0\right),\qquad I_4=\sum_{l=1}^{k-2}(\mathcal{L}w_l,\theta_0).$$

We next estimate $I_i, i\le 4$, respectively. Note that the function $E^xu(\cdot,y)\in C^1(\cdot,y)$ is continuous about y and there hold (see [12])

$$\partial_x^n E^xv(x_i,y)=0,\quad \partial^2_{xx}E^xv\perp\mathbb{P}_{k-2}(x),\quad \partial_y^nE^xv(x,y)=E^x\left(\partial_y^n v\right),\quad \forall n, \tag{5.34}$$
$$\|E^xv\|_{0,m}+h\left\|\partial_xE^xv\right\|_{0,m}+h^2\left\|\partial^2_{xx}E^xv\right\|_{0,m}\lesssim h^l\|v\|_{l,m},\qquad l\le k+1. \tag{5.35}$$

Then

$$|I_2|=\left|\left(\mathcal{L}E^xu,\theta_0\right)\right|=\left|\left(\mathcal{L}_2E^xu,\theta_0\right)\right|\le\begin{cases}h^{k+1}\|u\|_{k+2}\|\theta\|_0, & \text{if}\quad k=3,\\ 0, & \text{if}\quad k\ge 4.\end{cases}$$

By the approximation property of interpolation function $\mathcal{I}^x_{2k-3},\mathcal{I}^y_{2k-3}$, we have

$$|I_1|\lesssim h^{2k-2}\|\mathcal{L}E^xu\|_{2k-2}\|\theta_0\|_0\lesssim h^{2k-2}\|u\|_{2k}\|\theta_0\|_0.$$

Similarly, in light of (5.31), we get

$$|I_3|\lesssim h^{2k-2}\left\|\partial_y^{2k-2}\mathcal{L}w_l\right\|_{0,\infty}\|\theta_0\|_0\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta_0\|_0.$$

As for I_4, we use the integration by parts, (5.29), (5.28), and the fact that $J_n\perp\mathbb{P}_0(x), n\ge 5$ to obtain that

$$\begin{aligned}|I_4|=\left|\sum_{l=1}^{k-2}(\mathcal{L}_2w_l,\theta_0)\right|&\lesssim\sum_{l=1}^{k-1}\sum_{i=1}^{M}\left(h\left\|\mathcal{L}_2c^l_{i,1}\right\|_{0,\infty}+h^2\left\|\mathcal{L}_2c^l_{i,2}\right\|_{0,\infty}\right)\|\theta\|_0\\&\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta_0\|_0.\end{aligned}$$

We combine all the estimates of $I_i, 1\le i\le 4$ together and then get

$$\left|a\left(E^xu+w_h^x,\theta_0\right)\right|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta_0\|_0.$$

Consequently,

$$a\left(E^xu+w_h^x,\theta\right)=a\left(E^xu+w_h^x,\theta_0+\theta_1\right)\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta\|_0,\quad\forall\theta\in W_h.$$

This finishes the proof of (5.33). The proof of Proposition 5.1 is complete. □

5.3.2 Construction of the function w_h^y for $a(E^y u,\theta)$

Following the same argument, we can define a sequence of function $\bar{\omega}_l \in V_h, 1\le 1\le l\le k-2$ on the element band B_j^y. Noticing that $\partial_{yy}\bar{\omega}_l|_{B_j^y}\in\mathbb{P}_{k-2}(y)$, we define $\bar{\omega}_l$ as follows:

$$\partial_{yy}\bar{\omega}_l|_{B_j^y}=\sum_{p=1}^{k-2} d_{j,p}^l(x)L_{j,p}(y),\quad \partial_y\bar{\omega}_l(x,y_j)=0,\quad j\in\mathbb{Z}_N,\quad \bar{\omega}_l(x,c)=0,\quad \forall x\in[a,b], \tag{5.36}$$

where $d_{j,p}^l(x)\in\mathbb{P}_k(x)$ is the solution of the following equation:

$$-\sum_{i=1}^{M}(v,\theta)_{*,\tau_i^x}=\sum_{i=1}^{M}(\bar{\zeta},\theta)_{*,\tau_i^x},\quad \forall\theta\in\mathbb{P}_{k-2}\left(\tau_i^x\right),\quad v(a)=v(b)=0 \tag{5.37}$$

with

$$\bar{\zeta}=\bar{\zeta}_{j,p}^l(y):=\begin{cases}\dfrac{2p+1}{\alpha h_j^y}\left(\mathcal{L}E^y u,L_{j,p}\right)_{*,\tau_j^y}, & \text{if}\quad l=1,\\ \dfrac{2p+1}{\alpha h_j^y}\left((\alpha\partial_{xx}+\mathbf{fi}\cdot\nabla+\gamma)\bar{\omega}_{l-1},L_{j,p}\right)_{*,\tau_j^y}, & \text{if}\quad 1<l\le k-2.\end{cases}$$

Define

$$w_h^y(x,y):=\sum_{l=1}^{k-2}\bar{\omega}_l(x,y). \tag{5.38}$$

By the same argument as what we did in Lemma 5.3 and Proposition 5.1, we have

$$|\bar{\omega}_l(x_i,y_j)|+|\nabla\bar{\omega}_l(x_i,y_j)|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}, \tag{5.39}$$

$$\|\nabla\bar{\omega}_l\|_{0,\infty}\lesssim h^{k+l}\|u\|_{2k-1,\infty},\quad \left\|\partial_x^n\bar{\omega}_l\right\|_{0,\infty}\lesssim h^m\left\|\partial_x^n u\right\|_{m,\infty}, \tag{5.40}$$

$$\left|a\left(E^y u+w_h^y,\theta\right)\right|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta\|_0 \tag{5.41}$$

with $m\le\min(k+l+1,2k-2)$.

5.3.3 Construction of w_h and proof of Theorem 5.1

Now we define the correction function $w_h\in V_h^0$ as follows:

$$w_h(x,y)=\left(w_h^x+w_h^y\right)(x,y)-\frac{x-a}{b-a}w_h^x(b,y)-\frac{y-c}{d-c}w_h^y(x,d),$$

where w_h^x,w_h^y are defined by (5.32) and (5.38).

We are ready to prove Theorem 5.1.

Proof. First, by the first equation of (5.29) and the definition of w_h^x in (5.32) and (5.25), we get

$$w_h^x(x,c)=w_h^x(x,d)=w_h^x(a,y)=0,\quad \forall x\in[a,b],\quad y\in[c,d].$$

Similarly, there holds

$$w_h^y(a,y)=w_h^y(b,y)=w_h^x(x,c)=0,\quad \forall x\in[a,b],\quad y\in[c,d].$$

Then

$$w_h(x,y)=0,\quad \forall (x,y)\in\partial\Omega.$$

Second, in light of (5.25), (5.27)-(5.28) and the estimates of $c_{i,1}^l$ in (5.29), we have

$$\left|\mathcal{L}w_h^x(b,y)\right|=\left|\frac{h_M^x}{2}\sum_{l=1}^{k-1}\mathcal{L}_2c_{M,1}^l(y)\int_a^b\phi_{M,2}(x)dx\right|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}.$$

Following the same argument, there holds

$$\left|\mathcal{L}w_h^y(x,d)\right|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}.$$

Consequently, by denoting

$$\tilde{w}_h(x,y)=\frac{x-a}{b-a}w_h^x(b,y)+\frac{y-c}{d-c}w_h^y(x,d),$$

we have

$$\|\mathcal{L}\tilde{w}_h\|_{0,\infty}\lesssim\left|\mathcal{L}w_h^x(b,y)\right|+\left|\mathcal{L}w_h^y(x,d)\right|\lesssim h^{2k-2}\|u\|_{2k+1,\infty}.$$

Then for all $\theta\in W_h$,

$$|a(\tilde{w}_h,\theta)|\lesssim\|\mathcal{L}\tilde{w}_h\|_{0,\infty}\|\theta\|_0\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta\|_0,$$

which yields, together with (5.33), (5.41) and the estimate of E^xE^yu in (5.19) that

$$\begin{aligned}|a(u-P_hu+w_h,\theta)|&=\left|a\left(E^xu+w_h^x,\theta\right)+a\left(E^yu+w_h^y,\theta\right)-a(E^yE^xu,\theta)-a(\tilde{w}_h,\theta)\right|\\&\lesssim h^{2k-2}\|u\|_{2k+1,\infty}\|\theta\|_0.\end{aligned}$$

The proof is complete. □

6 Numerical experiments

In this section, we shall present some numerical examples to verify our theoretical findings in previous sections. The C^1-conforming Gauss collocation method is adopted for solving the convection-diffusion equation (2.1) with $k=3,4,5$. Non-uniform meshes of $M\times N$ rectangles are obtained by randomly and independently perturbing each node in the x- and y- axes of a uniform mesh as

$$\begin{aligned}x_i&=\frac{i}{M}+\varepsilon\frac{1}{M}\sin\left(\frac{i\pi}{M}\right)\text{randn}(),\quad 0\le i\le M,\\ y_j&=\frac{j}{N}+\varepsilon\frac{1}{N}\sin\left(\frac{j\pi}{N}\right)\text{randn}(),\quad 0\le j\le N,\end{aligned}$$

where randn() returns a uniformly distributed random number in $(0,1)$. For simplicity, we always choose $M=N$ and $\varepsilon=0.001$ in the following experiments.

We shall measure various errors between the exact solution u and the numerical solution u_h as defined in Theorem 5.2, including $e_{u,n}$ (i.e. the function value error at mesh nodes), $e_{\nabla u,n}$ (i.e. the gradient value error at mesh nodes), $e_{u,J}$ (i.e. the function value error on roots of $J_{k+1}^{-2,-2}(x)J_{k+1}^{-2,-2}(y)$), $e_{\nabla u,l}$ (i.e. the gradient value error on the Lobotto lines), and $e_{\Delta u,g}$ (i.e. the second-order derivative error on the Gauss lines). Errors between u_h and the special truncated Jacobi projection P_hu in all L^2,H^1,H^2-norms are also presented.

Example 6.1. We consider the problem (2.1) in $\Omega=(0,1)\times(0,1)$ with the following constant coefficients:

$$\alpha=\gamma=1, \quad \beta=(1,1).$$

The right-hand side function $f(x,y)$ is chosen such that the exact solution is

$$u(x,y)=xy(1-x)(1-y)e^{x+y}.$$

To test the superconvergence phenomena of u_h, we present in Table 1 various approximation errors of $u-u_h$ for $k=3,4,5$. We observe that both convergence rates of the function value error (i.e. $e_{u,n}$) and the gradient value error (i.e. $e_{\nabla u,n}$) at mesh nodes are $\mathcal{O}(h^{2k-2})$. Moreover, the convergence rates of average errors $e_{u,J},e_{\nabla u,l}$ and $e_{\Delta u,g}$ can reach $\mathcal{O}(h^{\min(k+2,2k-2)}),\mathcal{O}(h^{k+1})$, and $\mathcal{O}(h^k)$, respectively. All these numerical results are consistent with the theoretical results established in (5.14)-(5.16).

Table 1: Errors, corresponding convergence rates of $u-u_h$ for $k=3,4,5$ in Example 6.1.

k	M	$e_{u,n}$		$e_{\nabla u,n}$		$e_{u,J}$		$e_{\nabla u,l}$		$e_{\Delta u,g}$	
		Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
3	2	8.60e-04	-	2.09e-03	-	-	-	2.17e-03	-	1.32e-02	-
	4	4.21e-05	4.35	1.11e-04	4.23	-	-	1.47e-04	3.88	1.40e-03	3.24
	8	2.28e-06	4.22	7.66e-06	3.86	-	-	9.93e-06	3.89	1.67e-04	3.07
	16	1.33e-07	4.09	5.21e-07	3.87	-	-	6.48e-07	3.93	2.06e-05	3.01
4	2	5.13e-06	-	3.00e-05	-	3.22e-06	-	8.79e-05	-	6.10e-04	-
	4	6.17e-08	6.39	4.37e-07	6.11	4.93e-08	6.04	3.11e-06	4.83	3.18e-05	4.27
	8	8.33e-10	6.21	6.69e-09	6.03	7.71e-10	6.00	8.87e-08	5.13	1.94e-06	4.04
	16	1.21e-11	6.11	1.04e-10	6.01	1.21e-11	6.01	2.70e-09	5.05	1.18e-07	4.04
5	2	3.12e-08	-	1.27e-07	-	6.26e-08	-	2.33e-06	-	1.90e-05	-
	4	1.21e-10	8.01	4.98e-10	8.00	4.82e-10	7.03	3.43e-08	6.09	5.41e-07	5.14
	8	4.70e-13	8.01	1.96e-12	7.98	3.75e-12	7.00	5.62e-10	5.93	1.66e-08	5.03
	16	1.75e-15	8.08	7.78e-15	8.00	2.92e-14	7.02	8.82e-12	6.00	5.18e-10	5.01

Table 2: Errors, corresponding convergence rates of $u_h - P_h u$ for $k=3,4,5$ in Example 6.1.

k	M	$\\|u_h - P_h u\\|_0$		$\\|u_h - P_h u\\|_1$		$\\|u_h - P_h u\\|_2$	
		Error	Order	Error	Order	Error	Order
3	2	3.04e-04	-	2.14e-03	-	2.34e-02	-
	4	2.87e-05	3.40	1.79e-04	3.57	3.16e-03	2.89
	8	1.93e-06	3.90	1.19e-05	3.92	4.16e-04	2.93
	16	1.23e-07	3.96	7.58e-07	3.97	5.27e-05	2.98
4	2	4.03e-06	-	2.26e-05	-	2.33e-04	-
	4	7.62e-08	5.73	5.62e-07	5.34	1.23e-05	4.25
	8	1.25e-09	5.93	1.48e-08	5.24	7.29e-07	4.08
	16	1.99e-11	5.99	4.37e-10	5.09	4.47e-08	4.03
5	2	2.80e-08	-	2.81e-07	-	7.40e-06	-
	4	2.47e-10	6.82	4.89e-09	5.85	2.13e-07	5.12
	8	1.81e-12	7.09	8.37e-11	5.87	6.60e-09	5.01
	16	1.33e-14	7.10	1.35e-12	5.97	2.07e-10	5.01

Listed in Table 2 are the errors of $u_h - P_h u$ in all L^2, H^1 and H^2-norms. As we may observe, the convergence rates for errors $\|u_h - P_h u\|_0, \|u_h - P_h u\|_1$ and $\|u_h - P_h u\|_2$ are $\mathcal{O}(h^{\min(k+2,2k-2)}), \mathcal{O}(h^{k+1})$ and $\mathcal{O}(h^k)$, respectively. These results verify the convergence orders predicted in (5.13). In other words, the C^1-conforming Gauss collocation solution u_h is superconvergent towards the particular Jacobi projection $P_h u$ of the exact solution u.

Example 6.2. We consider the problem (2.1) in $\Omega = (0,1) \times (0,1)$ with variable coefficients of two cases

Case 1: Continuous α with $\alpha(x,y) = e^{xy}$.

Case 2: Piecewise continuous α with $\alpha(x,y) = \begin{cases} 1, & 0 \le x < 0.3, \quad 0 \le y \le 1, \\ 4, & 0.3 \le x \le 1, \quad 0 \le y \le 1. \end{cases}$

In both cases, the coefficients β, γ are taken as

$$\beta(x,y) = (x^2 y, xy^2), \quad \gamma(x,y) = 2xy.$$

The right-hand side function $f(x,y)$ is chosen such that the exact solution is

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Presented in Tables 3 and 4 are various errors and corresponding convergence rates of $u - u_h$ with $k=3,4,5$ in Cases 1 and 2, respectively. Just the same as that for the constant coefficient problem in Example 6.1, we observe a convergence order of $\mathcal{O}(h^{2k-2})$ for the errors $e_{u,n}$ and $e_{\nabla u,n}$, and a convergence order of $\mathcal{O}(h^{\min(k+2,2k-2)})$ for $e_{u,J}$, and $\mathcal{O}(h^{k+1})$ for $e_{\nabla u,l}$, and $\mathcal{O}(h^k)$ for $e_{\Delta u,g}$, which indicates that all the theoretical findings in (5.14)-(5.16) are also valid for variable coefficient problems and piecewise constant coefficient problems.

Table 3: Errors, corresponding convergence rates of $u-u_h$ in Example 6.2 (Case 1) with $k=3,4,5$.

k	M	$e_{u,n}$		$e_{\nabla u,n}$		$e_{u,J}$		$e_{\nabla u,l}$		$e_{\Delta u,g}$	
		Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
3	2	1.17e-02	-	6.63e-03	-	-	-	1.46e-02	-	5.95e-02	-
	4	4.74e-04	4.63	8.05e-04	3.05	-	-	9.33e-04	3.98	4.42e-03	3.76
	8	2.51e-05	4.24	4.97e-05	4.02	-	-	5.59e-05	4.06	4.79e-04	3.21
	16	1.46e-06	4.11	3.08e-06	4.01	-	-	3.51e-06	4.00	5.88e-05	3.03
4	2	1.48e-05	-	1.10e-06	-	1.20e-04	-	1.11e-03	-	4.32e-03	-
	4	4.06e-07	5.19	3.13e-06	-1.51	2.11e-06	5.83	3.56e-05	4.96	2.09e-04	4.37
	8	6.17e-09	6.03	5.26e-08	5.89	3.39e-08	5.95	1.18e-06	4.91	1.07e-05	4.2
	16	9.29e-11	6.07	8.41e-10	5.98	5.33e-10	6.00	3.51e-08	5.08	6.47e-07	4.06
5	2	1.11e-06	-	9.01e-07	-	2.69e-06	-	5.46e-05	-	2.59e-04	-
	4	2.71e-09	8.68	5.87e-09	7.26	2.19e-08	6.94	8.59e-07	5.99	6.70e-06	5.27
	8	8.87e-12	8.26	2.38e-11	7.95	1.73e-10	6.99	1.57e-08	5.78	1.98e-07	5.09
	16	3.25e-14	8.11	9.34e-14	8.00	1.36e-12	7.01	2.46e-10	6.01	6.07e-09	5.03

Table 4: Errors, corresponding convergence rates of $u-u_h$ in Example 6.2 (Case 2) with $k=3,4,5$.

k	M	$e_{u,n}$		$e_{\nabla u,n}$		$e_{u,J}$		$e_{\nabla u,l}$		$e_{\Delta u,g}$	
		Error	Order	Error	Order	Error	Order	Error	Order	error	Order
3	2	1.16e-02	-	5.41e-04	-	-	-	4.33e-02	-	4.44e-01	-
	4	4.74e-04	4.62	7.65e-04	-0.50	-	-	2.44e-03	4.15	4.96e-02	3.16
	8	2.51e-05	4.23	4.79e-05	3.99	-	-	1.47e-04	4.05	5.52e-03	3.17
	16	1.46e-06	4.11	2.97e-06	4.02	-	-	8.45e-06	4.13	6.62e-04	3.06
4	2	1.10e-05	-	6.72e-06	-	1.25e-04	-	3.39e-03	-	5.15e-02	-
	4	3.68e-07	4.90	3.08e-06	1.13	3.68e-06	5.09	1.26e-04	4.76	3.62e-03	3.83
	8	5.71e-09	6.01	5.14e-08	5.90	6.67e-08	5.78	4.14e-06	4.93	2.42e-04	3.91
	16	8.56e-11	6.06	8.19e-10	5.97	1.07e-09	5.97	1.37e-07	4.92	1.53e-05	3.98
5	2	1.18e-06	-	7.80e-08	-	4.10e-06	-	1.97e-04	-	3.37e-03	-
	4	2.77e-09	8.72	3.01e-09	4.69	3.08e-08	7.05	2.92e-06	6.07	8.59e-05	5.29
	8	9.10e-12	8.26	1.47e-11	7.68	2.46e-10	6.97	4.13e-08	6.15	2.55e-06	5.08
	16	3.30e-14	8.12	7.56e-14	7.62	1.90e-12	7.03	6.36e-10	6.03	7.62e-08	5.07

Table 5: Errors, corresponding convergence rates of $u_h - P_h u$ in Example 6.2 (Case 1) with $k=3,4,5$.

k	M	$\|u_h - P_h u\|_0$		$\|u_h - P_h u\|_1$		$\|u_h - P_h u\|_2$	
		Error	Order	Error	Order	Error	Order
3	2	4.34e-03	-	2.35e-02	-	1.79e-01	-
	4	3.26e-04	3.74	1.71e-03	3.78	2.52e-02	2.83
	8	2.15e-05	3.92	1.12e-04	3.94	3.26e-03	2.95
	16	1.36e-06	3.98	7.05e-06	3.99	4.10e-04	2.99
4	2	9.91e-05	-	6.68e-04	-	8.24e-03	-
	4	2.00e-06	5.63	2.06e-05	5.02	5.30e-04	3.96
	8	3.41e-08	5.87	6.47e-07	4.99	3.34e-05	3.98
	16	5.45e-10	5.98	2.02e-08	5.01	2.10e-06	4.00
5	2	2.56e-06	-	3.30e-05	-	6.57e-04	-
	4	2.13e-08	6.90	5.67e-07	5.86	2.17e-05	4.92
	8	1.73e-10	6.95	9.17e-09	5.96	6.97e-07	4.96
	16	1.36e-12	7.00	1.44e-10	6.00	2.19e-08	5.00

Table 6: Errors, corresponding convergence rates of $u_h - P_h u$ in Example 6.2 (Case 2) with $k=3,4,5$.

k	M	$\|u_h - P_h u\|_0$		$\|u_h - P_h u\|_1$		$\|u_h - P_h u\|_2$	
		Error	Order	Error	Order	Error	Order
3	2	4.41e-03	-	2.36e-02	-	1.72e-01	-
	4	3.29e-04	3.74	1.69e-03	3.80	2.32e-02	2.89
	8	2.16e-05	3.93	1.10e-04	3.95	2.96e-03	2.97
	16	1.36e-06	3.99	6.89e-06	4.00	3.70e-04	3.00
4	2	1.02e-04	-	6.85e-04	-	8.39e-03	-
	4	2.01e-06	5.68	2.05e-05	5.07	5.18e-04	4.02
	8	3.37e-08	5.90	6.34e-07	5.01	3.26e-05	3.99
	16	5.35e-10	5.98	1.97e-08	5.01	2.04e-06	4.00
5	2	2.38e-06	-	3.11e-05	-	6.22e-04	-
	4	2.01e-08	6.88	5.35e-07	5.86	2.04e-05	4.93
	8	1.63e-10	6.96	8.63e-09	5.96	6.55e-07	4.97
	16	1.28e-12	7.00	1.36e-10	6.00	2.06e-08	5.00

To demonstrate the supercloseness between u_h and $P_h u$ for variable coefficient problems, we present in Tables 5 and 6 the errors of $u_h - P_h u$ under L^2, H^1, and H^2-norms. Again, we see that the convergence rates for $\|u_h - P_h u\|_m, 0 \le m \le 2$ are $\mathcal{O}(h^{\min(k+2,2k-2)})$, $\mathcal{O}(h^{k+1}), \mathcal{O}(h^k)$, respectively. These results are consistent with the theoretical findings in (5.13).

7 Conclusion

In this work, we have studied superconvergence properties of C^1-conforming Gauss collocation methods for two-dimensional elliptic equations. A unified approach has been presented to prove that: The C^1 Gauss collocation solution is $(2k-2)$-th order of superconvergence in the function value and first-order derivative value approximations at mesh nodes; and $(k+2)$-th order of superconvergence at roots of the Jacobi polynomial $J_{k+1}^{-2,-2}(x)\otimes J_{k+1}^{-2,-2}(y)$, and $(k+1)$-th order of superconvergence in the first-order derivative at Lobatto lines; and k-th order of superconvergence in the second-order derivative at Gauss lines. An unexpected discovery is that the superconvergence of the first-order derivative at mesh points can reach as high as $2k-2$, which almost doubles the optimal convergence rate k. The superconvergence points for the second-order derivative approximation are novel. As we may recall, all the superconvergence results are similar to these for the counterpart C^1 Petrov-Galerkin method.

Comparing with the traditional C^0 Galerkin method, the major gain of the C^1 Gauss collocation method discussed in this work is the $(2k-2)$-th convergence rate in the first-order approximation at nodes and k-th convergence rate in the second-order approximation at Gauss lines, with the sacrifice of function value convergence rate at nodes dropping from $2k$ to $2k-2$.

Appendix A

In this section, we give the proofs of some lemmas used in the paper.

A.1 Proof of Lemma 3.1

Proof. Let

$$I_\tau = \int_\tau v v_{xxyy} dxdy - \sum_{m,n=1}^{k-1} (v v_{xxyy})\left(g_{m,n}^\tau\right) w_{m,n}^\tau.$$

Denote $G_i, w_i, i \le k-1$ be the $k-1$ Gauss points and the wights in $[-1,1]$. By the error of Gauss numerical quadrature (see, e.g. [20, p. 98 (2.7.12)], we have

$$\int_{-1}^1 f(s)ds - \sum_{i=1}^{k-1} f(G_i) w_i = \frac{2^{2k-1}[(k-1)!]^4}{(2k-1)[(2k-2)!]^3} \partial_s^{2k-2} f(\theta) = c_k \partial_s^{2k-2} f(\theta),$$

where $\theta \in (-1,1)$ and

$$c_k = \frac{2^{2k-1}[(k-1)!]^4}{(2k-1)[(2k-2)!]^3}. \tag{A.1}$$

Then

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1} f(s,t)dsdt - \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} f(G_i,G_j)w_iw_j \\
=&\int_{-1}^{1}\left(\int_{-1}^{1} f(s,t)ds - \sum_{i=1}^{k-1} f(G_i,t)w_i\right)dt + \sum_{i=1}^{k-1} w_i\left(\int_{-1}^{1} f(G_i,t)dt - \sum_{j=1}^{k-1} f(G_i,G_j)w_j\right) \\
=&c_k\int_{-1}^{1}\partial_s^{2k-2} f(\theta,t)dt + c_k\int_{-1}^{1}\partial_t^{2k-2} f(s,\eta)ds - (c_k)^2\partial_s^{2k-2}\partial_t^{2k-2} f(\zeta,\eta),
\end{aligned}$$

where θ,ζ,η are some points in $(-1,1)$. Consequently, by scaling from $(-1,1)$ to $\tau_i^x := (x_{i-1},x_i)$ and $\tau_j^y := (y_{j-1},y_j)$, there exist some $\zeta_i,\theta_i \in (x_{i-1},x_i), \eta_j \in (y_{j-1},y_j)$ such that for all $\tau = \tau_{i,j}$

$$\begin{aligned}
I_\tau =& \int_\tau (vv_{xxyy})(x,y)dxdy - \sum_{m=1}^{k-1}\sum_{n=1}^{k-1}(vv_{xxyy})\left(g_{m,n}^\tau\right)w_{m,n}^\tau \\
=& c_i^x \int_{y_{j-1}}^{y_j}\partial_x^{2k-2}(vv_{xxyy})(\theta_i,y)dy + c_j^y\int_{x_{i-1}}^{x_i}\partial_y^{2k-2}(vv_{xxyy})(x,\eta_j)dx - \bar{I}_{i,j},
\end{aligned}$$

where

$$\begin{aligned}
&c_i^x = c_k\left(\frac{x_i - x_{i-1}}{2}\right)^{2k-1}, \quad c_j^y = c_k\left(\frac{y_j - y_{j-1}}{2}\right)^{2k-1}, \\
&\bar{I}_{i,j} = c_i^x c_j^y \frac{\partial^{4k-4}(vv_{xxyy})}{\partial_x^{2k-2}\partial_y^{2k-2}}(\zeta_i,\eta_j) = c_i^x c_j^y\left(\frac{\partial^{2k} v}{\partial x^k \partial y^k}\right)^2(\zeta_i,\eta_j) \geq 0.
\end{aligned} \tag{A.2}$$

Consequently,

$$\begin{aligned}
I(v) =& \sum_{i=1}^{M}\sum_{j=1}^{N} I_{\tau_{i,j}} \leq \sum_{i=1}^{M} c_i^x \int_c^d \partial_x^{2k-2}(vv_{xxyy})(\theta_i,y)dy + \sum_{j=1}^{N} c_j^y\int_a^b \partial_y^{2k-2}(vv_{xxyy})(x,\eta_j)dx \\
=& -\sum_{i=1}^{M} c_i^x\int_c^d\left(\frac{\partial^{k+1}v}{\partial x^k\partial y}\right)^2(\theta_i,y)dy - \sum_{j=1}^{N} c_j^y\int_a^b\left(\frac{\partial^{k+1}v}{\partial x\partial y^k}\right)^2(x,\eta_j)dx \leq 0.
\end{aligned}$$

Here in the second step, we have used the integration by parts. Similarly, let

$$J_\tau = \int_\tau \triangle(vv_{xxyy})dxdy - \sum_{m,n=1}^{k-1}(\triangle vv_{xxyy})\left(g_{m,n}^\tau\right)w_{m,n}^\tau.$$

By using the error of Gauss numerical quadrature in (A.2), and the equation

$$\partial_x^{2k-2}\partial_y^{2k-2}(vv_{xxyy}) = 0,$$

and a scaling from $(-1,1)$ to $\tau_i^x := (x_{i-1}, x_i)$ and $\tau_j^y := (y_{j-1}, y_j)$ again, we get for all $\tau = \tau_{i,j}$,

$$\begin{aligned} J_\tau &= c_i^x \int_{y_{j-1}}^{y_j} \partial_x^{2k-2} (\triangle v v_{xxyy})(\theta_i, y) dy + c_j^y \int_{x_{i-1}}^{x_i} \partial_y^{2k-2} (\triangle v v_{xxyy})(x, \eta_j) dx \\ &= c_i^x \int_{y_{j-1}}^{y_j} \left(\partial_x^k \partial_y^2 v(\theta_i, y)\right)^2 dy + \int_{x_{i-1}}^{x_i} \left(\partial_y^k \partial_x^2 v(x, \eta_j) dx\right)^2 dx \geq 0 \end{aligned}$$

with $\theta_i \in \tau_i^x, \eta_j \in \tau_j^y$ being some constants. Consequently,

$$J(v) = \sum_{\tau \in \mathcal{T}_h} J_\tau \geq 0.$$

As for the Gauss numerical quadrature error $E(v)$, we use (A.2) again to obtain

$$\begin{aligned} E(v) &= \sum_{\tau_{i,j} \in \mathcal{T}_h} \left(c_i^x \beta_2 \int_{y_{j-1}}^{y_j} \left(\partial_x^k \partial_y^2 v\right) \left(\partial_x^k \partial_y v\right) (\theta_i, y) dy + c_j^y \beta_1 \int_{x_{i-1}}^{x_i} \left(\partial_y^k \partial_x^2 v\right) \left(\partial_y^k \partial_x v\right) (x, \eta_j) dx \right) \\ &= \sum_{\tau_{i,j} \in \mathcal{T}_h} \left(\frac{c_i^x \beta_2}{2} \int_{y_{j-1}}^{y_j} \partial_y \left(\partial_x^k \partial_y v\right)^2 (\theta_i, y) dy + \frac{c_j^y \beta_1}{2} \int_{x_{i-1}}^{x_i} \partial_x \left(\partial_y^k \partial_x v\right)^2 (x, \eta_j) dx \right) \\ &= \sum_{i=1}^{M} \frac{c_i^x \beta_2}{2} \left(\partial_x^k \partial_y v\right)^2 (\theta_i, y) \Big|_{y=c}^{d} + \sum_{j=1}^{N} \frac{c_j^y \beta_1}{2} \left(\partial_y^k \partial_x v\right)^2 (x, \eta_j) \Big|_{x=a}^{b}, \end{aligned}$$

where c_i^x, c_j^y are the same as that in (A.2), and $f(s)|_{s=s_1}^{s_2} = f(s_2) - f(s_1)$. By using the inverse inequality, we have

$$\left| \left(\partial_x^k \partial_y v\right)^2 (\theta_i, c) \right| \lesssim h^{-2} \int_{x_{i-1}}^{x_i} \int_c^d \left(\partial_x^k \partial_y v\right)^2 dx dy \lesssim h^{-2k+2} \int_{x_{i-1}}^{x_i} \int_c^d \left(\partial_x^2 \partial_y v\right)^2 dx dy.$$

Similarly, there holds

$$\left| \left(\partial_y^k \partial_x v\right)^2 (a, \eta_j) \right| \lesssim h^{-2} \int_{y_{j-1}}^{y_j} \int_a^b \left(\partial_y^k \partial_x v\right)^2 dx dy \lesssim h^{-2k+2} \int_{y_{j-1}}^{y_j} \int_a^b \left(\partial_y^2 \partial_x v\right)^2 dx dy.$$

Substituting the above two inequalities into the formula of $E(v)$ yields

$$|E(v)| \lesssim h \left(\|v_{xyy}\|_0^2 + \|v_{xxy}\|_0^2 \right).$$

The proof is complete. □

A.2 Proof of Lemma 5.3

Proof. We only prove (5.29) for $n=0$. The same argument can be applied to any positive n. First, by (5.26) and the fact that $(k-1)$-point Guass numerical quadrature is exact for polynomial of degree $2k-3$, we have

$$\begin{aligned}\frac{\alpha h_i^x}{(2p+1)}\zeta_{i,p}^1(y) &= \left(\mathcal{I}_{2k-3-p}^x \mathcal{L} E^x u, L_{i,p}\right)_{*,\tau_i^x} = \int_{x_{i-1}}^{x_i} \left(\mathcal{I}_{2k-3-p}^x \mathcal{L} E^x u\right)(x,y) L_{i,p}(x)dx \\ &= \int_{x_{i-1}}^{x_i} \mathcal{L} E^x u(x,y) L_{i,p}(x)dx + \int_{x_{i-1}}^{x_i} \left(\mathcal{I}_{2k-3-p}^x \mathcal{L} E^x u - \mathcal{L} E^x u\right)(x,y) L_{i,p}(x)dx \\ &= J_1 + I_1.\end{aligned}$$

As for J_1, we have, from the integration by parts and the fact that $\partial_{xx} E^x u \perp \mathbb{P}_{k-2}(x)$

$$J_1 = \int_{x_{i-1}}^{x_i} \mathcal{L}_2 E^x u(x,y) L_{i,p}(x)dx - \frac{h_i^x}{2}\int_{x_{i-1}}^{x_i} \beta_1 E^x u(x,y)\phi_{i,p+1}(x)dx.$$

Using the estimate of $E^x u$ in (5.34)-(5.35) and the fact that $E^x u \perp \mathbb{P}_{k-4}(x)$, there holds

$$\|J_1\|_{0,\infty} \lesssim \begin{cases} h\left\|\mathcal{L}_2 E^x u\right\|_{0,\infty} + h^2\|E^x u\|_{0,\infty}, & \text{if} \quad k-3 \le p \le k-2, \\ h^2\|E^x u\|_{0,\infty}, & \text{if} \quad p = k-4, \\ 0, & \text{if} \quad p < k-4. \end{cases}$$

On the other hand, by the approximation property of the interpolation function,

$$\|I_1\|_{0,\infty} \le h\left\|\mathcal{I}_{2k-3-p}^x \mathcal{L} E^x u - \mathcal{L} E^x u\right\|_{0,\infty} \lesssim h^{m+1}\|\mathcal{L} E^x u\|_{m,\infty}, \quad k \le m \le 2k-2-p.$$

Consequently,

$$\left\|c_{i,p}^1\right\|_{0,\infty} \lesssim \left\|\zeta_{i,p}^1\right\|_{0,\infty} \lesssim h^{-1}(\|J_1\|_{0,\infty} + \|I_1\|_{0,\infty}) \lesssim h^m\|u\|_{m+2,\infty}$$

for $k \le m \le 2k-2-p$. Noticing that $2k-2-p \ge k$, then (5.29) holds true for $l=1$. This finishes the proof of (5.29) for $k=3$.

When $k \ge 4$, we use the method of mathematical induction to prove (5.29). We first suppose (5.29) is valid for all $i \le l-1$ and then show that it also holds for l with $2 \le l \le k-2$. By (5.26) and the error of Gauss numerical quadrature, we have

$$\frac{\alpha h_i^x \zeta_{i,p}^l}{2p+1} = \int_{x_{i-1}}^{x_i} (\mathcal{L}_1 w_{l-1})(x,y) L_{i,p}(x)dx - c_k\left(\frac{h_i^x}{2}\right)^{2k-1} \partial_x^{2k-2}(\mathcal{L}_1 w_{l-1} L_{i,p})(\rho_i, y) = J + I,$$

where c_k is given in (A.1), and ρ_i is some point in τ_i^x. We next estimate I and J, respectively. Since $\mathcal{L}_1 w_{l-1} L_{i,p} \in \mathbb{P}_{k-2}(x)$ for all $p \le k-2$ and $\partial_x^m L_{i,p} = \mathcal{O}(h^{-m})$, we have

$$\|I\|_{0,\infty} \lesssim h^{2k-1}\left\|\partial_x^k(\mathcal{L}_1 w_{l-1})\partial_x^{k-2} L_{i,p}\right\|_{0,\infty} \lesssim \begin{cases} h^3\left\|\mathcal{L}_2 c_{i,k-2}^{l-1}\right\|_{0,\infty}, & \text{if} \quad p = k-2, \\ 0, & \text{if} \quad p < k-2. \end{cases}$$

When it comes to J, we have that, from (5.27) and the integration by parts,

$$\begin{aligned}
J &= \int_{x_{i-1}}^{x_i} (\mathcal{L}_1 w_{l-1})(x,y) L_{i,p}(x) dx \\
&= \frac{h_i^x}{2} \int_{x_{i-1}}^{x_i} \partial_x (\mathcal{L}_2 w_{l-1})(x,y) \phi_{i,p+1}(x) dx + \beta_1 \int_{x_{i-1}}^{x_i} \partial_x w_{l-1}(x,y) L_{i,p}(x) dx \\
&= \left(\frac{h_i^x}{2}\right)^2 \sum_{q=1}^{k-2} \mathcal{L}_2 c_{i,q}^{l-1} \int_{x_{i-1}}^{x_i} (\phi_{i,q+1}\phi_{i,p+1})(x) dx + \beta_1 \frac{h_i^x}{2} \sum_{q=1}^{k-2} c_{i,q}^{l-1} \int_{x_{i-1}}^{x_i} (\phi_{i,q+1} L_{i,p})(x) dx.
\end{aligned}$$

By the properties of Legendre and Lobatto polynomials, we get

$$\|J\|_{0,\infty} \lesssim h^3 \sum_{q=p,p+2,p-2} \big\|\mathcal{L}_2 c_{i,q}^{l-1}\big\|_{0,\infty} + h^2 \sum_{q=p-1,p+1} \big\|c_{i,q}^{l-1}\big\|_{0,\infty}, \tag{A.3}$$

and thus

$$\begin{aligned}
\big\|c_{i,p}^{l}\big\|_{0,\infty} &\lesssim h^{-1}(\|J\|_{0,\infty} + \|I\|_{0,\infty}) \\
&\lesssim h^2 \sum_{q=p,p+2,p-2} \big\|\mathcal{L}_2 c_{i,q}^{l-1}\big\|_{0,\infty} + h \sum_{q=p-1,p+1} \big\|c_{i,q}^{l-1}\big\|_{0,\infty} + h^2 \big\|c_{i,k-2}^{l-1}\big\|_{0,\infty} \delta_{p,k-2},
\end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta with value 1 when $i=j$ and 0 otherwise.

Noticing that $\mu_{l-1,p-1} \geq \mu_{l-1,p+1}$, by the inductive hypothesis, there holds for any $m_1 \in [0,\mu_{l-1,p-1}], m_2 \in [0,\mu_{l-1,p+1}]$ that

$$\begin{aligned}
h \sum_{q=p-1,p+1} \big\|c_{i,q}^{l-1}\big\|_{0,\infty} &\lesssim h^{m_1+1} \|u\|_{m_1+2,\infty} + h^{m_2+1} \|u\|_{m_2+2,\infty} \\
&\lesssim h^{m_2+1} \|u\|_{m_2+2,\infty} = h^{m'} \|u\|_{m'+1,\infty},
\end{aligned} \tag{A.4}$$

where

$$m' = m_2 + 1 \leq 1 + \mu_{l-1,p+1} = 1 + \max(2k-p-3, k+l-2) = \mu_{l,p}.$$

Similarly, there holds for any $m_1 \in [0,\mu_{l-1,p-2}], m_2 \in [0,\mu_{l-1,p+2}]$ such that

$$\begin{aligned}
h^2 \sum_{q=p-2,p+2} \big\|\mathcal{L}_2 c_{i,q}^{l-1}\big\|_{0,\infty} &\lesssim h^{m_1+2} \|\mathcal{L}_2 u\|_{m_1+2,\infty} + h^{m_2+2} \|\mathcal{L}_2 u\|_{m_2+2,\infty} \\
&\lesssim h^{m_2+2} \|u\|_{m_2+4,\infty} = h^{m'} \|u\|_{m'+2,\infty},
\end{aligned} \tag{A.5}$$

where

$$m' = m_2 + 2 \leq 2 + \mu_{l-1,p+2} = \max(2k-p-2, k+l) \in [\mu_{l,p}, \mu_{l,p}+1].$$

Following the same arguments, there holds for any $m \leq \mu_{l-1,p}$ such that

$$h^2 \big\|\mathcal{L}_2 c_{i,p}^{l-1}\big\|_{0,\infty} \lesssim h^{m+2} \|\mathcal{L}_2 u\|_{m+2,\infty} \lesssim h^{m+2} \|u\|_{m+4,\infty} \leq h^{m'} \|u\|_{m'+2,\infty} \tag{A.6}$$

with

$$m' = m_2 + 2 \leq 2 + \mu_{l-1,p} = \max(2k-p, k+l) \in [\mu_{l,p}, \mu_{l,p-1}+1].$$

Combining (A.4)-(A.6) together yields

$$\|c^l_{i,p}\|_{0,\infty} \lesssim h^m \|u\|_{m+2,\infty}, \quad \forall p \in \mathbb{Z}_{k-2}, \quad m \leq \mu_{l,p}.$$

In other words, (5.29) is also valid for l. This completes proof of the induction for $k \geq 4$.

By (5.27)-(5.29), we have

$$\partial_x w_l(x_i, y_j) = 0, \quad |\partial^n_y w_l(x_i, y_j)| = \left|\frac{h^x_i}{2} \partial^n_y c^l_{i,1}(y_j)\right| \lesssim h^{2k-2} \|u\|_{2k,\infty}, \quad n = 0, 1.$$

Then (5.30) follows. By choosing $m = \mu_{l,p}$ in (5.29) and using (5.27)-(5.28), we obtain the desired result (5.31) immediately. The proof is complete. $\square$

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