Numer. Math. Theor. Meth. Appl. Vol. **xx**, No. x, pp. 1-32
doi: 10.4208/nmtma.OA-2023-0048 x 2024

An Accurate Numerical Scheme for Mean-Field Forward and Backward SDEs with Jumps

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Received 21 April 2023; Accepted (in revised version) 27 November 2023

Abstract. In this work, we propose an explicit second order scheme for decoupled mean-field forward backward stochastic differential equations with jumps. The stability and the rigorous error estimates are presented, which show that the proposed scheme yields a second order rate of convergence, when the forward mean-field stochastic differential equation is solved by the weak order 2.0 Itô-Taylor scheme. Numerical experiments are carried out to verify the theoretical results.

AMS subject classifications: 65C30, 60H10, 60H35, 65C05

Key words: Mean-field forward backward stochastic differential equation with jumps, stability analysis, error estimates.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ being the filtration generated by the following two mutually independent stochastic processes:

- The m-dimensional Brownian motion $W = (W_t)_{0\le t\le T}$.
- The Poisson random measure $\{\mu(A \times [0,t]),\ A \in \mathcal{E},\ 0 \le t \le T\}$ on $\mathsf{E} \times [0,T]$, where $\mathsf{E} = \mathbb{R}^q\backslash\{0\}$ and $\mathcal{E}$ is its Borel field.

In this paper, we suppose that the Poisson measure μ has the intensity measure

$$\nu(de, dt) = \lambda(de)dt = \lambda F(de)dt,$$

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where $\lambda(de)$ is a Lévy measure on $(\mathsf{E},\mathcal{E})$ describing the average number of jumps per unit of time, $\lambda=\lambda(\mathsf{E})$ is the intensity of the measure μ and F is the distribution of the jump size e. Here $\lambda(de)$ is a σ-finite measure satisfying

$$\int_{\mathsf{E}}\left(1\wedge|e|^2\right)\lambda(de)<+\infty.$$

Moreover, we have the compensated Poisson random measure

$$\tilde{\mu}(de,dt)=\mu(de,dt)-\lambda(de)dt,$$

such that $\{\tilde{\mu}(A\times[0,t])=(\mu-\nu)(A\times[0,t])\}_{0\le t\le T}$ is a martingale for any $A\in\mathcal{E}$.

The Poisson measure μ can generate a sequence of pairs $(\tau_i,e_i),i=1,2,\ldots,N_T$ with $\tau_i\in[0,T],i=1,2,\ldots,N_T$, representing the jump times of N_t and $e_i\in\mathsf{E},i=1,2,\ldots,N_T$ the corresponding jump sizes satisfying $e_i\overset{iid}{\sim}F$. Here $N_t=\mu(\mathsf{E}\times[0,t])$ is a Poisson process with intensity λ, which counts the number of jumps of μ occurring in $[0,t]$. For more details of the Poisson random measure and Lévy measure, the readers are referred to [6,17].

We are interested in the following general mean-field forward backward stochastic differential equations with jumps (MFBSDEJs for short) on $(\Omega,\mathcal{F},\mathbb{F},P)$

$$\begin{aligned}
X_t^{0,X_0}&=X_0+\int_0^t\mathbb{E}\big[b\big(s,X_s^{0,x_0},x\big)\big]\big|_{x=X_s^{0,X_0}}ds\\
&\quad+\int_0^t\mathbb{E}\big[\sigma\big(s,X_s^{0,x_0},x\big)\big]\big|_{x=X_s^{0,X_0}}dW_s\\
&\quad+\int_0^t\int_{\mathsf{E}}\mathbb{E}\big[c\big(s,X_{s-}^{0,x_0},x,e\big)\big]\big|_{x=X_{s-}^{0,X_0}}\tilde{\mu}(de,ds),\\
Y_t^{0,X_0}&=\mathbb{E}\big[\Phi\big(X_T^{0,x_0},x\big)\big]\big|_{x=X_T^{0,X_0}}\\
&\quad+\int_t^T\mathbb{E}\big[f\big(s,\Theta_s^{0,x_0},\theta\big)\big]\big|_{\theta=\Theta_s^{0,X_0}}ds\\
&\quad-\int_t^T Z_s^{0,X_0}dW_s-\int_t^T\int_{\mathsf{E}}U_s^{0,X_0}(e)\tilde{\mu}(de,ds),
\end{aligned}\tag{1.1}$$

where

$$\Theta_s^{0,x}=\big(X_s^{0,x},Y_s^{0,x},Z_s^{0,x},\Gamma_s^{0,x}\big)$$

with $x=x_0$ and X_0 being the initial values of mean-field forward stochastic differential equations with jumps (MSDEJs). Here, $\Gamma_s^{0,x}$ is defined by

$$\Gamma_s^{0,x}=\int_{\mathsf{E}}U_s^{0,x}(e)\eta(e)\lambda(de)$$

for a given function $\eta:\mathsf{E}\to\mathbb{R}$ satisfying $\sup_{e\in\mathsf{E}}|\eta(e)|<+\infty$,

$$\begin{aligned}
&b:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d,\\
&\sigma:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^{d\times m},\\
&c:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathsf{E}\to\mathbb{R}^d
\end{aligned}$$

are respectively drift, diffusion, and jump coefficients of MSDEJs, $\mathbb{E}[\Phi(X_T^{0,x_0}, x)]|_{x=X_T^{0,X_0}}$ is the terminal condition with $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^p$, and

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \times \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \times \mathbb{R}^p \to \mathbb{R}^p$$

is the generator of mean-field backward stochastic differential equations with jumps (MBSDEJs). A quadruplet $(X_t^{0,X_0}, Y_t^{0,X_0}, Z_t^{0,X_0}, U_t^{0,X_0})$ is called an L^2-adapted solution of (1.1), if it is $\mathcal{F}_t$-adapted, square integrable and satisfies (1.1). In general, the initial values X_0 and x_0 are different, and $(X_t^{0,x_0}, Y_t^{0,x_0}, Z_t^{0,x_0}, U_t^{0,x_0})$ is the solution of (1.1) with $X_0 = x_0$.

The theory of mean-field forward backward stochastic differential equations (MFBSDEs for short) was initially developed by Buckdahn *et al.* [3] in 2009. Since then, MFBSDEJs have become an important tool in many research areas such as the nonlocal diffusion problems [2–4, 11], stochastic optimal control [7, 13, 25, 26], and mean-field games [1, 5, 8, 12]. Furthermore, Li [14] extended the theory of MFBSDEs to the framework of MFBSDEJs. MFBSDEJs can obviously model the event-driven stochastic phenomena much more accurately by comprising Lévy jump processes, and hence admit much wider applications in the above research areas [10, 15, 16, 21, 24]. In view of its wide applications, it is important and interesting to study numerical methods for solving MFBSDEJs. Due to the Poisson random measure and the nonlocal properties of MFBSDEJs, it is very difficult to construct numerical methods for MFBSDEJs.

To prepare for the numerical methods for MFBSDEJs, we developed the Itô's formula and Itô-Taylor expansion for mean-field SDEs and SDEJs, and constructed general Itô-Taylor schemes for them in our previous works [18, 21]. Then the authors presented high order θ-schemes for MBSDEs in [22]. Furthermore, a second order one-step scheme and a third order multi-step scheme were proposed in [19, 20] for solving decoupled MFBSDEs. To our knowledge, nevertheless, there are few works on MFBSDEJs in the literature.

In this paper, we are devoted to numerical methods for solving decoupled MFBSDEJs. By solving MSDEJs with the Itô-Taylor schemes proposed in the paper [21], we will design a second order numerical scheme for solving decoupled MFBSDEJs. By using the Itô formula in mean-field version, we will first rigorously analyze the stability of the proposed scheme, and then derive its error estimates from the obtained stability results. The error estimates show that the proposed scheme admits a first order convergence rate when MSDEJs are solved by the Euler scheme or the Milstein scheme, and a second order convergence rate when MSDEJs are solved by the weak order 2.0 Itô-Taylor scheme. Our numerical results show that the proposed scheme is stable, effective and can be of second order rate of convergence, which are consistent with our theoretical conclusions.

It is worth pointing out that compared with the numerical methods for solving FBSDEs (short for forward backward stochastic differential equations), the methods for mean-field FBSDEs with jumps are computationally expensive and complicated. The main reasons are listed as below.

- First, we need to approximate the expectations with respect to the solutions contained in the coefficients of mean-field FBSDEs with jumps. Since the probability density functions of the solutions are unknown, it is not easy to approximate these expectations efficiently.

- Second, our constructed Scheme 3.1 contains several conditional expectations with respect to the solutions. It can be very time-consuming and complicated to approximate these conditional expectations because of the existing of the discrete Poisson jumps in the solutions, whose numbers and sizes are both random variables.

To overcome the above two difficulties, we first apply the Monte-Carlo method to simulate the expectations contained in the coefficients of MFBSDEJs. As for the conditional expectations in our scheme, we write them in the form of multiple integrals by using the distributions of the Brownian motion, the jump numbers and the jump sizes, and the independence of these random variables. Then we approximate the corresponding integrals by using the high-efficient Gaussian quadrature rules. For more details, please refer to Section 5.

The paper is organized as follows. In Section 2, we present some preliminaries including Itô's formula and the Feynman-Kac formula. In Section 3, by discretizing MFBSDEJs in time, we develop an explicit second order numerical scheme for MFBSDEJs. Stability analysis and error estimates are performed in Section 4. In Section 5, some numerical experiments are carried out to verify our theoretical results, and we finally conclude the paper in Section 6.

We close this section by listing some notation that will be used in what follows:

- $|\cdot|$: the standard Euclidean norm in the Euclidean space.

- $C_b^{2,2}$: the set of continuous differential functions $\phi(x,y)$ with uniformly bounded partial derivatives $\partial_x^{k_1}\partial_y^{k_2}\phi$ for $k_1 \leq 2$ and $k_2 \leq 2$.

- $C_b^{1,2,2}$: the set of continuous differential functions $\phi(t,x,y)$ with uniformly bounded partial derivatives $\partial_t^{l_1}\phi$ and $\partial_y^{k_1}\partial_z^{k_2}\phi$ for $l_1 \leq 1$ and $k_1+k_2 \leq 2$. Moreover, we can define $C_b^{1,2,2,2,2,2,2,2,2}$ in a similar way.

2. Preliminaries

In this section, we will introduce some useful results including the nonlinear Feynman-Kac formula and Itô's formula for general MSDEJs.

2.1. The nonlinear Feynman-Kac formula

To show the representations of the solutions of decoupled MFBSDEJs, we recall the nonlinear Feynman-Kac formula in this subsection, which explains why we can numerically solve the MFBSDEJs (1.1) in spatiotemporal framework in this paper.

For simplicity, we make the following assumption on the coefficients and the terminal condition.

Assumption 2.1. Assume that $b, \sigma \in C_b^{1,2,2}$ and $c(\cdot,\cdot,\cdot,e) \in C_b^{1,2,2}$ with the bound of $K(1 \wedge |e|)$ for all its derivatives of first and second order, $f \in C_b^{1,2,2,2,2,2,2,2,2,2}$ and $\Phi \in C_b^{2,2}$.

Now we present the following nonlinear Feynman-Kac formula [14].

Lemma 2.1. *Under Assumption* 2.1*, the solutions of the MBSDEJs in* (1.1) *have the following representations:*

$$\begin{aligned} Y_t^{0,X_0} &= u\big(t, X_t^{0,X_0}\big), \\ Z_t^{0,X_0} &= \nabla_x u\big(t, X_t^{0,X_0}\big)\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big]\big|_{x=X_t^{0,X_0}}, \\ U_t^{0,X_0} &= u\Big(t, X_{t-}^{0,X_0} + \mathbb{E}\big[c\big(t, X_{t-}^{0,x_0}, x, e\big)\big]\big|_{x=X_{t-}^{0,X_0}}\Big) - u\big(t-, X_{t-}^{0,X_0}\big), \end{aligned} \tag{2.1}$$

where $u(t,x)$ is the classical solution of the following nonlocal quasi-linear PIDE:

$$\begin{aligned} \mathcal{A}[u](t,x) + \mathbb{E}\Big[f\Big(&t, X_t^{0,x_0}, u\big(t, X_t^{0,x_0}\big), \\ &\nabla_x u\big(t, X_t^{0,x_0}\big)\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big]\big|_{x=X_t^{0,x_0}}, \\ &\mathcal{B}[u]\big(t-, X_{t-}^{0,x_0}\big), x, u(t,x), \\ &\nabla_x u(t,x)\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big], \mathcal{B}[u](t,x)\Big)\Big] = 0 \end{aligned} \tag{2.2}$$

with the terminal condition $u(T,x) = \mathbb{E}[\Phi(X_T^{0,x_0}, x)]$. Here $\mathcal{A}$ is a second order integral-differential operator defined as

$$\begin{aligned} \mathcal{A}[u](t,x) = &\frac{\partial u}{\partial t}(t,x) + \sum_{i=1}^{d} \mathbb{E}\Big[b_i\big(t, X_t^{0,x_0}, x\big)\Big]\frac{\partial u}{\partial x_i}(t,x) \\ &+ \frac{1}{2}\sum_{i,j=1}^{d}\Big(\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big]\mathbb{E}\big[\sigma\big(t, X_t^{0,x_0}, x\big)\big]^{\top}\Big)_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) \\ &+ \int_{\mathsf{E}}\Big(u\big(t, x + \mathbb{E}\big[c\big(t, X_{t-}^{0,x_0}, x, e\big)\big]\big) - u(t,x) \\ &\qquad - \sum_{i=1}^{d}\mathbb{E}\big[c_i\big(t, X_{t-}^{0,x_0}, x, e\big)\big]\frac{\partial u}{\partial x_i}(t,x)\Big)\lambda(de), \end{aligned}$$

and $\mathcal{B}$ is an integral operator defined as

$$\mathcal{B}[u](t,x) = \int_{\mathsf{E}}\Big(u\big(t, x + \mathbb{E}\big[c\big(t, X_{t-}^{0,x_0}, x, e\big)\big]\big) - u(t,x)\Big)\eta(e)\lambda(de). \tag{2.3}$$

Note that

$$\Gamma_t^{0,x} = \int_{\mathsf{E}} U_t^{0,x}(e)\eta(e)\lambda(de),$$

then by (2.1) and (2.3), we get

$$\Gamma_t^{0,X_0} = \mathcal{B}[u]\big(t-, X_{t-}^{0,X_0}\big).$$

Lemma 2.2. *It is known that when the functions b, σ, c, f and Φ are bounded and smooth enough with bounded derivatives, the PIDE* (2.2) *has a unique solution $u(t,x)$ which is also bounded and smooth with bounded derivatives* [14].

2.2. Itô's formula for MSDEJs

Let β_t be a d-dimensional Itô process with jumps defined by

$$d\beta_t = \psi_t dt + \varphi_t dW_t + \int_{\mathsf{E}} h_{t-}(e)\mu(de, dt), \tag{2.4}$$

where ψ_t, φ_t and h_t are progressively measurable processes satisfying

$$\int_0^T |\psi_t|\, dt < +\infty, \quad \int_0^T \mathrm{Tr}\big[\varphi_s \varphi_s^\top\big] dt < +\infty, \quad \int_0^T \int_{\mathsf{E}} |h_t(e)|^2\, \lambda(de)dt < +\infty, \quad \text{a.e..}$$

For notational simplicity, for two given functions $g_1 : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $g_2 : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathsf{E} \to \mathbb{R}$, we define

$$\begin{aligned}
&g_1^\beta(t,x) = \mathbb{E}\big[g_1\big(t, \beta_t, x\big)\big],\\
&g_2^\beta(t,x,e) = \mathbb{E}\big[g_2\big(t, \beta_t, x, e\big)\big],\\
&g_2^\beta(t-,x,e) = \mathbb{E}\big[g_2\big(t-, \beta_{t-}, x, e\big)\big].
\end{aligned}$$

Consider the following general MSDEJ:

$$dX_t = b^\beta(t, X_t)dt + \sigma^\beta(t, X_t)dW_t + \int_{\mathsf{E}} c^\beta(t-, X_{t-}, e)\mu(de, dt). \tag{2.5}$$

Note that under Assumption 2.1, the MSDEJ (2.5) has a unique solution. Now we state the Itô's formula [21] for the MSDEJ (2.5) in the following theorem.

Theorem 2.1. *Let X_t be the unique solution of the MSDEJ* (2.5). *Then for $f \in C^{1,2,2}$, $f^\beta(t, X_t)$ is an Itô process with jumps satisfying*

$$\begin{aligned}
f^\beta(t, X_t) = f^\beta(0, X_0) &+ \int_0^t L^0 f^\beta(s, X_s)\, ds + \int_0^t \overrightarrow{L}^1 f^\beta(s, X_s)\, dW_s\\
&+ \int_0^t \int_{\mathsf{E}} L_e^{-1} f^\beta(s, X_{s-})\, \mu(de, ds),
\end{aligned} \tag{2.6}$$

where the operators $L^0, \overrightarrow{L}^1$ and L_e^{-1} are defined as

$$\begin{aligned} L^0 f^\beta(s,x) &= \frac{\partial f^\beta}{\partial s}(s,x) + \nabla_x f^\beta(s,x)\, b^\beta(s,x) \\ &\quad + \frac{1}{2}\mathrm{Tr}\left[f^\beta_{xx}(s,x)\big(\sigma^\beta(s,x)\big)\big(\sigma^\beta(s,x)\big)^\top \right], \\ \overrightarrow{L}^1 f^\beta(s,x) &= \big(L^1 f^\beta(s,x), \ldots, L^m f^\beta(s,x)\big), \\ L_e^{-1} f^\beta(s,x) &= f^\beta\big(s, x + c^\beta(s-,x,e)\big) - f^\beta(s-,x) \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} &\frac{\partial f^\beta}{\partial s}(s,x) = \mathbb{E}\left[\frac{\partial f}{\partial s}(s,\beta_s,x) + \nabla_{x'} f(s,\beta_s,x)\,\psi_s + \frac{1}{2}\mathrm{Tr}\left[f_{x'x'}(s,\beta_s,x)\,\varphi_s\varphi_s^\top \right]\right], \\ &\nabla_x f^\beta(s,x) = \mathbb{E}\big[\nabla_x f(s,\beta_s,x)\big], \quad f^\beta_{xx}(s,x) = \mathbb{E}\big[f_{xx}(s,\beta_s,x)\big], \\ &L^j f^\beta(t,x) = \sum_{k=1}^d \frac{\partial f^\beta}{\partial x^k}(t,x)\,\sigma^\beta_{kj}(t,x), \quad j = 1,2,\ldots,m. \end{aligned}$$

Here σ_j denotes the j-th column of the matrix σ and

$$\nabla_x f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right), \quad f_{xx} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{d\times d}.$$

We remark that the above Itô's formula for MSDEJs will play an important role in the numerical analysis of our scheme. For the details of the Itô's formula (2.6), the readers are referred to [21].

3. Numerical scheme for MFBSDEJs

In this section, we first introduce the general Itô-Taylor schemes for solving MSDEJs, then based on which, we develop an explicit second order numerical scheme for solving the MFBSDEJs (1.1). For notational simplicity, we let $d = m = p = 1$.

Let N be a finite positive integer. For the temporal partition, we introduce a regular time partition on $[0,T]$

$$\mathcal{T} := \big\{ 0 = t_0 < t_1 < \cdots < t_N = T \big\}.$$

For the above regular time partition, we let

$$\Delta t_n = t_{n+1} - t_n, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n}, \quad \Delta N_n = N_{t_{n+1}} - N_{t_n}.$$

Here the regularity means there exists a constant $c_0 \geq 1$ (independent of N) such that

$$\frac{\max_{0\leq n\leq N-1} \Delta t_n}{\min_{0\leq n\leq N-1} \Delta t_n} \leq c_0. \tag{3.1}$$

3.1. The general Itô-Taylor schemes for MSDEJs

Let $X_n^{X_0}$ ($X_n^{x_0}$) be the approximation values of the solutions X_t^{0,X_0} (X_t^{0,x_0}) of the MSDEJ in (1.1) at time $t = t_n$ ($n = 0, 1, \ldots, N$), solved by a Itô-Taylor scheme proposed in [21] in the form

$$X_{n+1}^{X_0} = X_n^{X_0} + \mathbb{E}\left[\varphi\big(t_n, \Delta t_n, X_n^{x_0}, x, w, m, \tau, e\big)\right] \tag{3.2}$$

with $x = X_n^{X_0}$, $w = \Delta W_n$, $m = \Delta N_n$, $\tau = \boldsymbol{\tau}$ and $e = \mathbf{e}$, where φ is a method dependent function, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{\Delta N_n})$ and $\boldsymbol{e} = (e_1, \ldots, e_{\Delta N_n})$ with ΔN_n the jump number and (τ_i, e_i) the pairs of jump time and jump size occurring in $(t_n, t_{n+1}]$.

Define

$$\tilde{b}\left(t, x', x\right) = b\left(t, x', x\right) - \int_{\mathsf{E}} c\left(t, x', x, e\right) \lambda(de). \tag{3.3}$$

Then by taking different forms of the function φ (depends on b, σ, c and their derivatives), we give three examples of the Itô-Taylor scheme (3.2), see [18].

1. The Euler scheme

$$\begin{aligned} X_{n+1}^{X_0} = {} & X_n^{X_0} + \tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta W_n \\ & + \sum_{i=1}^{\Delta N_n} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_i\right). \end{aligned} \tag{3.4}$$

2. The Milstein scheme

$$\begin{aligned} X_{n+1}^{X_0} = {} & X_n^{X_0} + \tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\Delta W_n \\ & + \sum_{i=1}^{\Delta N_n} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_i\right) + \frac{1}{2}L^1\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\left((\Delta W_n)^2 - \Delta t_n\right) \\ & + \sum_{i=1}^{\Delta N_n} L^1 c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_i\right)\left(W_{\tau_i} - W_{t_n}\right) \\ & + \sum_{i=1}^{\Delta N_n} L_{e_i}^{-1}\sigma^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)\left(W_{t_{n+1}} - W_{\tau_i}\right) \\ & + \sum_{i=1}^{\Delta N_n} \sum_{j=N_{t_n}+1}^{N_{\tau_i -}} L_{e_j}^{-1} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_i\right). \end{aligned} \tag{3.5}$$

3. The weak order 2.0 Itô-Taylor scheme

$$\begin{aligned} X_{n+1}^{X_0} = {} & X_n^{X_0} + \tilde{b}^{X_n^{x_0}}\big(t_n, X_n^{X_0}\big)\Delta t_n + \sigma^{X_n^{x_0}}\big(t_n, X_n^{X_0}\big)\Delta W_n \\ & + \sum_{i=1}^{\Delta N_n} c^{X_n^{x_0}}\left(t_n, X_n^{X_0}, e_i\right) + \frac{1}{2}L^0\tilde{b}^{X_n^{x_0}}\left(t_n, X_n^{X_0}\right)(\Delta t_n)^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}L^1\sigma^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)\left((\Delta W_n)^2-\Delta t_n\right)\\
&+\sum_{i=1}^{\Delta N_n}L^1c^{X_n^{x_0}}\left(t_n,X_n^{X_0},e_i\right)\left(W_{\tau_i}-W_{t_n}\right)\\
&+\sum_{i=1}^{\Delta N_n}L_{e_i}^{-1}\sigma^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)\left(W_{t_{n+1}}-W_{\tau_i}\right)\\
&+\sum_{i=1}^{\Delta N_n}\sum_{j=N_{t_n}+1}^{N_{\tau_i-}}L_{e_j}^{-1}c^{X_n^{x_0}}\left(t_n,X_n^{X_0},e_i\right)\\
&+\sum_{i=1}^{\Delta N_n}L^0c^{X_n^{x_0}}\left(t_n,X_n^{X_0},e_i\right)(\tau_i-t_n)\\
&+\sum_{i=1}^{\Delta N_n}L_{e_i}^{-1}\tilde{b}^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)(t_{n+1}-\tau_i)\\
&+\frac{1}{2}\left(L^1\tilde{b}^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)+L^0\sigma^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)\right)\Delta W_n\Delta t_n.
\end{aligned}
\tag{3.6}
$$

Remark 3.1. Note that to solve the MSDEJ in (1.1) for $X_0 \neq x_0$, we need two steps in succession. We take the Euler scheme (3.4) for instance to illustrate this procedure.

- Step 1: solve the MSDEJ with $X_0 = x_0$ to obtain $\{X_n^{x_0}\}_{n=0}^N$

$$
\begin{aligned}
X_{n+1}^{x_0} = X_n^{x_0} &+ \tilde{b}^{X_n^{x_0}}\left(t_n,X_n^{x_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n,X_n^{x_0}\right)\Delta W_n\\
&+\sum_{i=1}^{\Delta N_n}c^{X_n^{x_0}}\left(t_n,X_n^{x_0},e_i\right).
\end{aligned}
$$

- Step 2: solve the MSDEJ with $X_0 \neq x_0$ to get $\{X_n^{X_0}\}_{n=0}^N$ after we get $\{X_n^{x_0}\}_{n=0}^N$

$$
\begin{aligned}
X_{n+1}^{X_0} = X_n^{X_0} &+ \tilde{b}^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)\Delta t_n + \sigma^{X_n^{x_0}}\left(t_n,X_n^{X_0}\right)\Delta W_n\\
&+\sum_{i=1}^{\Delta N_n}c^{X_n^{x_0}}\left(t_n,X_n^{X_0},e_i\right).
\end{aligned}
$$

Let C_p^k be the set of continuously differentiable functions $\phi(x)$ such that all its partial derivatives up to order k have a polynomial growth. Then we state some approximate properties of the Itô-Taylor scheme in (3.2) in the following proposition, which will be used in our error analysis.

Proposition 3.1. *Let $\{X_n^{X_0}, n = 0,\dots,N\}$ denote the numerical solutions of the Itô-Taylor scheme in* (3.2)*. Then there exist positive numbers $r_1, r_2, r_3, \alpha_1, \alpha_2$ and l such that for any $g \in C_P^{2l+2}$ and $n = 0, 1, \dots, N$,*

$$
\left|\mathbb{E}\left[g\left(X_{t_n}^{0,X_0}\right)-g\left(X_n^{X_0}\right)\right]\right| \le C_g(\Delta t)^l,
$$

$$\left|\mathbb{E}_{t_n}^{X_n}\left[g\big(X_{t_{n+1}}^{t_n,X_n^{X_0}}\big)-g\big(X_{n+1}^{X_0}\big)\right]\right|\le C_g\Big(1+\mathbb{E}\left[|X_n^{x_0}|^{2r_1}\right]+\left|X_n^{X_0}\right|^{2r_1}\Big)(\Delta t)^{l+1},$$

$$\left|\mathbb{E}_{t_n}^{X_n}\left[\Big(g\big(X_{t_{n+1}}^{t_n,X_n^{X_0}}\big)-g\big(X_{n+1}^{X_0}\big)\Big)\Delta\tilde{W}_n\right]\right|\le C_g\Big(1+\mathbb{E}\left[|X_n^{x_0}|^{2r_2}\right]+\left|X_n^{X_0}\right|^{2r_2}\Big)(\Delta t)^{\alpha_1+1},$$

$$\left|\mathbb{E}_{t_n}^{X_n}\left[\Big(g\big(X_{t_{n+1}}^{t_n,X_n^{X_0}}\big)-g\big(X_{n+1}^{X_0}\big)\Big)\Delta\tilde{\mu}_n^*\right]\right|\le C_g\Big(1+\mathbb{E}\left[|X_n^{x_0}|^{2r_3}\right]+\left|X_n^{X_0}\right|^{2r_3}\Big)(\Delta t)^{\alpha_2+1},$$

where C_g is a positive constant independent of Δt and l is called the global weak convergence order of the Itô-Taylor scheme in (3.2).

Remark 3.2. In Proposition 3.1, it holds that [21]:

1. $\alpha_1=\alpha_2=l=1$ for the Euler scheme (3.4) and the Milstein scheme (3.5).
2. $\alpha_1=\alpha_2=l=2$ for the weak order 2.0 Itô-Taylor scheme (3.6).

3.2. The explicit second order scheme for MFBSDEJs

Based on the nonlinear Feynman-Kac formula (2.1), we first discretize the MBSDEJ in (1.1) in time. Then by solving the MSDEJ in (1.1) using the Itô-Taylor scheme, we propose an explicit second order semi-discrete numerical scheme for solving the decoupled MFBSDEJs (1.1).

To derive the reference equations for constructing the numerical scheme, we define the following two stochastic processes $\Delta\tilde{W}_{t_n,s}$ and $\Delta\tilde{\mu}_{t_n,s}^*$ by

$$\Delta\tilde{W}_{t_n,s}=\int_{t_n}^{s}p(r)dW_r,\quad \Delta\tilde{\mu}_{t_n,s}^*=\int_{t_n}^{s}\int_{\mathsf{E}}p(r)\eta(e)\,\tilde{\mu}(de,dr)$$

for $t_n\le s\le T$, where $p(r)=2-(3(r-t_n))/\Delta t_n$. It is obvious that $\int_{t_n}^T p^2(r)dr<+\infty$, and thus the Itô integral $\Delta\tilde{W}_{t_n,s}$ is a martingale satisfying

$$\begin{aligned}\mathbb{E}_{t_n}^x\left[\Big(\Delta\tilde{W}_{t_n,s}\Big)^2\right]&=\mathbb{E}_{t_n}^x\left[\left(\int_{t_n}^s p(r)dW_r\right)^2\right]=\mathbb{E}_{t_n}^x\left[\int_{t_n}^s p^2(r)dr\right]\\&=\int_{t_n}^s\left(2-\frac{3(r-t_n)}{\Delta t_n}\right)^2dr\\&=(s-t_n)\left(1+\frac{3}{\Delta t_n^2}(s-t_{n+1})^2\right).\end{aligned}$$

Let $\Delta\tilde{W}_n=\Delta\tilde{W}_{t_n,t_{n+1}}$ and $\Delta\tilde{\mu}_n^*=\Delta\tilde{\mu}_{t_n,t_{n+1}}^*$. Then by taking $s=t_{n+1}$, we obtain

$$\mathbb{E}_{t_n}^x\big[\Delta\tilde{W}_n\big]=0,\quad \mathbb{E}_{t_n}^x\left[\big(\Delta\tilde{W}_n\big)^2\right]=\Delta t_n.$$

Since $\sup_{e\in\mathsf{E}}|\eta(e)|<+\infty$, it holds that

$$\int_{t_n}^T\int_{\mathsf{E}}p^2(r)\eta^2(e)\mu(de,dr)<+\infty.$$

Thus $\Delta\tilde{\mu}^*_{t_n,s}$ is a martingale satisfying

$$\mathbb{E}^x_{t_n}\left[\Delta\tilde{\mu}^*_n\right]=0,\quad \mathbb{E}^x_{t_n}\left[\left|\Delta\tilde{\mu}^*_n\right|^2\right]=\Delta t_n\int_{\mathsf{E}}\eta^2(e)\lambda(de).$$

Let $\Theta^{t,x}_s=(X^{t,x}_s,Y^{t,x}_s,Z^{t,x}_s,\Gamma^{t,x}_s)$ denote the unique solution of the MFBSDEJs (1.1) with the forward MSDEJ starting from the time-space point (t,x). Then for $n=N-1,\dots,1,0$, we have

$$\begin{aligned}Y^{t_n,x}_{t_n}=Y^{t_n,x}_{t_{n+1}}&+\int_{t_n}^{t_{n+1}}\mathbb{E}\left[f\big(s,\Theta^{0,x_0}_s,\theta\big)\right]|_{\theta=\Theta^{t_n,x}_s}ds\\&-\int_{t_n}^{t_{n+1}}Z^{t_n,x}_s dW_s-\int_{t_n}^{t_{n+1}}\int_{\mathsf{E}}U^{t_n,x}_s(e)\tilde{\mu}(de,ds).\end{aligned}\tag{3.7}$$

In the following, we first solve the unknowns $Z^{t_n,x}_{t_n}$ and $\Gamma^{t_n,x}_{t_n}$ based on (3.7). Using the obtained values of $Z^{t_n,x}_{t_n}$ and $\Gamma^{t_n,x}_{t_n}$, we solve $Y^{t_n,x}_{t_n}$ in an explicit way.

To solve $Z^{t_n,x}_{t_n}$, we multiply (3.7) with $\Delta\tilde{W}_n$ and take the conditional expectation

$$\mathbb{E}^x_{t_n}[\cdot]:=\mathbb{E}[\ \cdot\ |\mathcal{F}_{t_n},X^{0,X_0}_{t_n}=x]$$

on both sides of the derived equation to deduce

$$\begin{aligned}0=\mathbb{E}^x_{t_n}\left[Y^{t_n,x}_{t_{n+1}}\Delta\tilde{W}_n\right]&+\int_{t_n}^{t_{n+1}}\mathbb{E}^x_{t_n}\left[\mathbb{E}\left[f\big(s,\Theta^{0,x_0}_s,\theta\big)\right]|_{\theta=\Theta^{t_n,x}_s}\Delta\tilde{W}_n\right]ds\\-\mathbb{E}^x_{t_n}&\left[\int_{t_n}^{t_{n+1}}Z^{t_n,x}_s dW_s\cdot\Delta\tilde{W}_n\right].\end{aligned}$$

From the above equation, we get the reference equation for solving $Z^{t_n,x}_{t_n}$

$$\begin{aligned}\frac{1}{2}\Delta t_n Z^{t_n,x}_{t_n}=&\ \mathbb{E}^x_{t_n}\left[Y^{t_n,x}_{t_{n+1}}\Delta\tilde{W}_n\right]\\&+\Delta t_n\mathbb{E}^x_{t_n}\left[\mathbb{E}\left[f\big(t_{n+1},\Theta^{0,x_0}_{t_{n+1}},\theta\big)\right]\Big|_{\theta=\Theta^{t_n,x}_{t_{n+1}}}\Delta\tilde{W}_n\right]+R^{n,X_0}_z,\end{aligned}\tag{3.8}$$

where $R^{n,X_0}_z=R^{n,X_0}_{z_1}+R^{n,X_0}_{z_2}$ with

$$\begin{aligned}R^{n,X_0}_{z_1}=&\int_{t_n}^{t_{n+1}}\mathbb{E}^x_{t_n}\left[\mathbb{E}\left[f\big(s,\Theta^{0,x_0}_s,\theta\big)\right]|_{\theta=\Theta^{t_n,x}_s}\Delta\tilde{W}_n\right]ds\\&-\Delta t_n\mathbb{E}^x_{t_n}\left[\mathbb{E}\left[f\big(t_{n+1},\Theta^{0,x_0}_{t_{n+1}},\theta\big)\right]\Big|_{\theta=\Theta^{t_n,x}_{t_{n+1}}}\Delta\tilde{W}_n\right],\\R^{n,X_0}_{z_2}=&\ \frac{1}{2}\Delta t_n Z^{t_n,x}_{t_n}-\mathbb{E}^x_{t_n}\left[\int_{t_n}^{t_{n+1}}Z^{t_n,x}_s dW_s\cdot\Delta\tilde{W}_n\right].\end{aligned}$$

To solve $\Gamma_{t_n}^{t_n,x}$, we multiply (3.7) by $\Delta\tilde{\mu}_n^*$ and take $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the derived equation to obtain

$$
\begin{aligned}
0 = {} & \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\Delta\tilde{\mu}_n^*\right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(s,\Theta_s^{0,x_0},\theta\right)\right]\big|_{\theta=\Theta_s^{t_n,x}}\Delta\tilde{\mu}_n^*\right]ds \\
& - \mathbb{E}_{t_n}^x\left[\int_{t_n}^{t_{n+1}}\int_{\mathsf{E}} U_s^{t_n,x}(e)\tilde{\mu}(de,ds)\cdot\Delta\tilde{\mu}_n^*\right],
\end{aligned}
$$

from which, we get the reference equation for solving $\Gamma_{t_n}^{t_n,x}$

$$
\begin{aligned}
\frac{1}{2}\Delta t_n\Gamma_{t_n}^{t_n,x} = {} & \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\Delta\tilde{\mu}_n^*\right] \\
& + \Delta t_n\mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\right)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\Delta\tilde{\mu}_n^*\right] + R_\gamma^{n,X_0},
\end{aligned}
\tag{3.9}
$$

where $R_\gamma^{n,X_0} = R_{\gamma_1}^{n,X_0} + R_{\gamma_2}^{n,X_0}$ with

$$
\begin{aligned}
R_{\gamma_1}^{n,X_0} = {} & \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(s,\Theta_s^{0,x_0},\theta\right)\right]\big|_{\theta=\Theta_s^{t_n,x}}\Delta\tilde{\mu}_n^*\right]ds \\
& - \Delta t_n\mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\right)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\Delta\tilde{\mu}_n^*\right], \\
R_{\gamma_2}^{n,X_0} = {} & \frac{1}{2}\Delta t_n\Gamma_{t_n}^{t_n,x} - \mathbb{E}_{t_n}^x\left[\int_{t_n}^{t_{n+1}}\int_{\mathsf{E}} U_s^{t_n,x}\tilde{\mu}(de,ds)\cdot\Delta\tilde{\mu}_n^*\right].
\end{aligned}
$$

Now we consider the reference equation for solving $Y_{t_n}^{t_n,x}$. Using the fact that the stochastic integrals $\{\int_{t_n}^t Z_s^{t_n,x}dW_s\}_{t_n\le t\le T}$ and $\{\int_{t_n}^t\int_{\mathsf{E}} U_s^{t_n,x}(e)\tilde{\mu}(de,ds)\}_{t_n\le t\le T}$ are both martingales, we take $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of (3.7) to get

$$
\begin{aligned}
Y_{t_n}^{t_n,x} & = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(s,\Theta_s^{0,x_0},\theta\right)\right]\big|_{\theta=\Theta_s^{t_n,x}}\right]ds \\
& = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n\mathbb{E}\left[f\left(t_n,\Theta_{t_n}^{0,x_0},\theta\right)\right]\Big|_{\theta=\Theta_{t_n}^{t_n,x}} \\
& \quad + \frac{1}{2}\Delta t_n\mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\right)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right] + R_{y_1}^{n,X_0},
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
R_{y_1}^{n,X_0} = {} & \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(s,\Theta_s^{0,x_0},\theta\right)\right]\big|_{\theta=\Theta_s^{t_n,x}}\right]ds \\
& - \frac{1}{2}\Delta t_n\mathbb{E}\left[f\left(t_n,\Theta_{t_n}^{0,x_0},\theta\right)\right]\Big|_{\theta=\Theta_{t_n}^{t_n,x}} \\
& - \frac{1}{2}\Delta t_n\mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\right)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right].
\end{aligned}
$$

The expectations in (3.10) make it inefficient to solve $Y_{t_n}^{t_n,x}$ implicitly. To overcome this difficulty, in this paper, we will propose an explicit scheme for solving $Y_{t_n}^{t_n,x}$. To this end, we first present $Y_{t_n}^{t_n,x}$ in the form

$$Y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \Delta t_n \mathbb{E}_{t_n}^x\left[f\big(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\big)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}} + R_{yr}^{n,X_0},$$

where

$$\begin{aligned} R_{yr}^{n,X_0} = & \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\left(s,\Theta_s^{0,x_0},\theta\right)\right]\big|_{\theta=\Theta_s^{t_n,x}}\right] ds \\ & - \Delta t_n \mathbb{E}_{t_n}^x\left[f\big(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\big)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}. \end{aligned}$$

Define the prediction value $\bar{Y}_{t_n}^{t_n,x}$ of $Y_{t_n}^{t_n,x}$ by

$$\bar{Y}_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\big(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\big)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right], \tag{3.11}$$

and let

$$\begin{aligned} \bar{\Theta}_{t_n}^{t_n,x} &= \left(X_{t_n}^{t_n,x}, \bar{Y}_{t_n}^{t_n,x}, Z_{t_n}^{t_n,x}, \Gamma_{t_n}^{t_n,x}\right), \\ \bar{\Theta}_{t_n}^{0,x_0} &= \left(X_{t_n}^{0,x_0}, \bar{Y}_{t_n}^{0,x_0}, Z_{t_n}^{0,x_0}, \Gamma_{t_n}^{0,x_0}\right). \end{aligned}$$

We then get the following reference equation for solving $Y_{t_n}^{t_n,x}$:

$$\begin{aligned} Y_{t_n}^{t_n,x} = & \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n \mathbb{E}\left[f\big(t_n,\bar{\Theta}_{t_n}^{0,x_0},\theta\big)\right]\Big|_{\theta=\bar{\Theta}_{t_n}^{t_n,x}} \\ & + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[\mathbb{E}\left[f\big(t_{n+1},\Theta_{t_{n+1}}^{0,x_0},\theta\big)\right]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,x}}\right] + R_y^{n,X_0}, \end{aligned} \tag{3.12}$$

where $R_y^{n,X_0} = R_{y_1}^{n,X_0} + R_{y_2}^{n,X_0}$ with $R_{y_2}^{n,X_0}$ defined as

$$R_{y_2}^{n,X_0} = \Delta t_n\left(\mathbb{E}\left[f\big(t_n,\Theta_{t_n}^{0,x_0},\theta\big)\right]\Big|_{\theta=\Theta_{t_n}^{t_n,x}} - \mathbb{E}\left[f\big(t_n,\bar{\Theta}_{t_n}^{0,x_0},\theta\big)\right]\Big|_{\theta=\bar{\Theta}_{t_n}^{t_n,x}}\right).$$

Note that by the Feynman-Kac formula (2.1), the prediction values $\bar{Y}_{t_n}^{t_n,x}$ and $\bar{Y}_{t_n}^{0,x_0}$ are functions of (t_n,x) and $(t_n,X_{t_n}^{0,x_0})$, respectively, which can be interpreted as

$$\bar{Y}_{t_n}^{t_n,x} = \bar{Y}_{t_n}(x), \quad \bar{Y}_{t_n}^{0,x_0} = \bar{Y}_{t_n}\big(X_{t_n}^{0,x_0}\big).$$

Using the reference equations (3.8), (3.9), (3.11) and (3.12), we are ready to construct our explicit second order semi-discrete numerical scheme for solving the MFBSDEJs (1.1).

Let

$$\Theta_n^{X_0} = \big(X_n^{X_0}, Y_n^{X_0}, Z_n^{X_0}, \Gamma_n^{X_0}\big)$$

denote the numerical approximations of the solution $(X_t^{0,X_0}, Y_t^{0,X_0},\ Z_t^{0,X_0}, \Gamma_t^{0,X_0})$ of (1.1) at time $t = t_n$ and define

$$f_n^{x_0,X_0} = \mathbb{E}\big[f\left(t_n, \Theta_n^{x_0}, \theta\right)\big]\big|_{\theta=\Theta_n^{X_0}}, \quad n = 0, 1, \ldots, N.$$

Then by letting $x = X_n^{X_0}$ and removing the truncation error terms R_z^{n,X_0}, R_γ^{n,X_0}, R_{yr}^{n,X_0} and R_y^{n,X_0} in (3.8), (3.9), (3.11) and (3.12), we propose the following explicit second order scheme for solving (1.1).

Scheme 3.1. Step 1. Given initial value x_0, solve $X_n^{x_0}$ for $n = 1, \ldots, N$ by the Itô-Taylor scheme (3.2).

Step 2. Given initial value X_0, and terminal conditions $Y_N^{X_0}$, $Z_N^{X_0}$ and Γ_N^{0,X_0}, for $n = N-1, \ldots, 0$, we solve $Y_n^{X_0} = Y_n(X_n^{X_0})$, $Z_n^{X_0} = Z_n(X_n^{X_0})$ and $\Gamma_n^{X_0} = \Gamma_n(X_n^{X_0})$ by

$$\frac{1}{2}\Delta t_n Z_n^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\Delta\tilde{W}_n\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1}^{x_0,X_0}\Delta\tilde{W}_n\right], \tag{3.13}$$

$$\frac{1}{2}\Delta t_n \Gamma_n^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\Delta\tilde{\mu}_n^*\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1}^{x_0,X_0}\Delta\tilde{\mu}_n^*\right], \tag{3.14}$$

$$\bar{Y}_n^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1}^{x_0,X_0}\right], \tag{3.15}$$

$$Y_n^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1}^{X_0}\right] + \frac{1}{2}\Delta t_n \bar{f}_n^{x_0,X_0} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1}^{x_0,X_0}\right], \tag{3.16}$$

where $X_{n+1}^{X_0}$ is solved by the Itô-Taylor scheme (3.2), and

$$\bar{f}_n^{x_0,X_0} = \mathbb{E}\left[f\left(t_n, \bar{\Theta}_n^{x_0}, \theta\right)\right]\big|_{\theta=\bar{\Theta}_n^{X_0}}$$

with $\bar{\Theta}_n^x = (X_n^x, \bar{Y}_n^x, Z_n^x, \Gamma_n^x)$ for $x = x_0$ and X_0.

We remark that the terminal conditions used in Scheme 3.1 are given by

$$\begin{aligned}
Y_T^{0,X_0} &= \mathbb{E}\big[\Phi\big(X_T^{0,x_0}, x\big)\big]\big|_{x=X_T^{0,X_0}}, \\
Z_T^{0,X_0} &= \nabla_x \mathbb{E}\big[\Phi\big(X_T^{0,x_0}, x\big)\big]\big|_{x=X_T^{0,X_0}} \mathbb{E}\big[\sigma\big(T, X_T^{0,x_0}, x\big)\big]\big|_{x=X_T^{0,X_0}}, \\
\Gamma_T^{0,X_0} &= \int_E \Big(\mathbb{E}\Big[\Phi\Big(X_T^{0,x_0}, x + \mathbb{E}\big[c\big(T-, X_{T-}^{0,x_0}, x, e\big)\big]\Big)\Big]\Big|_{x=X_{T-}^{0,X_0}} \\
&\qquad - \mathbb{E}\big[\Phi\big(X_{T-}^{0,x_0}, x\big)\big]\big|_{x=X_{T-}^{0,X_0}}\Big)\eta(e)\lambda(de).
\end{aligned}$$

Remark 3.3. It is clear that Scheme 3.1 is explicit for solving Y_n^x, Z_n^x and Γ_n^x, calculating from the time level t_{n+1} to t_n. And the approximations of the conditional expectations in the scheme are presented in detail in Section 5.2.

We also remark that Scheme 3.1 can not be applied to solve general mean-field FBSDEJs whose coefficients depend on the probability distribution $\mathbb{P}_{X_s^{0,x_0}}$ of X_s^{0,x_0} in a nonlinear way as shown in [9, 23].

4. Stability analysis and error estimates

In this section, we first study the stability of Scheme 3.1, and then give its error estimates using the derived stability results. For simplicity, we only perform the analysis in the one-dimensional setting. But all the conclusions hereafter can be extended to multidimensional cases.

4.1. Stability analysis

To analyze the stability of Scheme 3.1, we define

$$\begin{aligned} &Y_{N,\varepsilon}^{X_0} = Y_N^{X_0} + \varepsilon_y^{N,X_0}, \\ &Z_{N,\varepsilon}^{X_0} = Z_N^{X_0} + \varepsilon_z^{N,X_0}, \\ &\Gamma_{N,\varepsilon}^{X_0} = \Gamma_N^{X_0} + \varepsilon_\gamma^{N,X_0}, \\ &f_\varepsilon = f + \varepsilon_f, \end{aligned}$$

where ε_f and $(\varepsilon_y^{N,X_0}, \varepsilon_z^{N,X_0}, \varepsilon_\gamma^{N,X_0})$ denote the random perturbations on the generator f and the terminal condition $(Y_N^{X_0}, Z_N^{X_0}, \Gamma_N^{X_0})$, respectively. Here we assume that

$$\varepsilon_f = \varepsilon_f \left(t, x', y', z', \gamma', x, y, z, \gamma\right)$$

is a $\mathcal{F}_t$-adapted stochastic process for any given $(t, x', y', z', \gamma', x, y, z, \gamma) \in [0,T] \times \mathbb{R}^8$. For notational simplicity, we let

$$\begin{aligned} f_{n,\varepsilon}^{x_0,X_0} &= \mathbb{E}\left[f\left(t_n, \Theta_{n,\varepsilon}^{x_0}, \theta\right)\right]\big|_{\theta=\Theta_{n,\varepsilon}^{X_0}}, \\ \bar{f}_{n,\varepsilon}^{x_0,X_0} &= \mathbb{E}\left[f\left(t_n, \bar{\Theta}_{n,\varepsilon}^{x_0}, \theta\right)\right]\big|_{\theta=\bar{\Theta}_{n,\varepsilon}^{X_0}}, \\ \varepsilon_{f,n}^{x_0,X_0} &= \mathbb{E}\left[\varepsilon_f\left(t_n, \Theta_{n,\varepsilon}^{x_0}, \theta\right)\right]\big|_{\theta=\Theta_{n,\varepsilon}^{X_0}}, \\ \bar{\varepsilon}_{f,n}^{x_0,X_0} &= \mathbb{E}\left[\varepsilon_f\left(t_n, \bar{\Theta}_{n,\varepsilon}^{x_0}, \theta\right)\right]\big|_{\theta=\bar{\Theta}_{n,\varepsilon}^{X_0}}, \end{aligned}$$

where

$$\begin{aligned} \Theta_{n,\varepsilon}^x &= \left(X_n^x, Y_{n,\varepsilon}^x, Z_{n,\varepsilon}^x, \Gamma_{n,\varepsilon}^x\right), \\ \bar{\Theta}_{n,\varepsilon}^x &= (X_n^x, \bar{Y}_{n,\varepsilon}^x, Z_{n,\varepsilon}^x, \Gamma_{n,\varepsilon}^x) \end{aligned}$$

for $x = x_0$ and X_0 with $\bar{Y}_{n,\varepsilon}^{X_0}$, $Y_{n,\varepsilon}^{X_0}$, $Z_{n,\varepsilon}^{X_0}$ and $\Gamma_{n,\varepsilon}^{X_0}$ being the solutions of Scheme 3.1 with perturbations on f and $(Y_N^{X_0}, Z_N^{X_0}, \Gamma_N^{X_0})$, which satisfy

$$\begin{aligned} &\frac{1}{2}\Delta t_n Z_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\Delta\tilde{W}_n\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\left(f_{n+1,\varepsilon}^{x_0,X_0} + \varepsilon_{f,n+1}^{x_0,X_0}\right)\Delta\tilde{W}_n\right], \\ &\frac{1}{2}\Delta t_n \Gamma_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\Delta\tilde{\mu}_n^*\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\left(f_{n+1,\varepsilon}^{x_0,X_0} + \varepsilon_{f,n+1}^{x_0,X_0}\right)\Delta\tilde{\mu}_n^*\right], \\ &\bar{Y}_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0} + \varepsilon_{f,n+1}^{x_0,X_0}\right], \end{aligned}$$

$$Y_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\right] + \frac{1}{2}\Delta t_n\left(\bar{f}_{n,\varepsilon}^{x_0,X_0} + \bar{\varepsilon}_{f,n}^{x_0,X_0}\right) + \frac{1}{2}\Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0} + \varepsilon_{f,n+1}^{x_0,X_0}\right],$$

or equivalently

$$\frac{1}{2}\Delta t_n Z_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\Delta\tilde{W}_n\right] + \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0}\Delta\tilde{W}_n\right] + R_{\varepsilon z}^{n,X_0}, \tag{4.1}$$

$$\frac{1}{2}\Delta t_n \Gamma_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\Delta\tilde{\mu}_n^*\right] + \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0}\Delta\tilde{\mu}_n^*\right] + R_{\varepsilon z}^{n,X_0}, \tag{4.2}$$

$$\bar{Y}_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\right] + \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0}\right] + \bar{R}_{\varepsilon y}^{n,X_0}, \tag{4.3}$$

$$Y_{n,\varepsilon}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{n+1,\varepsilon}^{X_0}\right] + \frac{1}{2}\Delta t_n\bar{f}_{n,\varepsilon}^{x_0,X_0} + \frac{1}{2}\Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0}\right] + R_{\varepsilon y}^{n,X_0}, \tag{4.4}$$

where $\bar{R}_{\varepsilon y}^{n,X_0}$, $R_{\varepsilon y}^{n,X_0}$, $R_{\varepsilon z}^{n,X_0}$ and $R_{\varepsilon\gamma}^{n,X_0}$ are the perturbation terms

$$\begin{aligned}
R_{\varepsilon z}^{n,X_0} &= \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_{f,n+1}^{x_0,X_0}\Delta\tilde{W}_n\right],\\
R_{\varepsilon\gamma}^{n,X_0} &= \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_{f,n+1}^{x_0,X_0}\Delta\tilde{\mu}_n^*\right],\\
\bar{R}_{\varepsilon y}^{n,X_0} &= \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_{f,n+1}^{x_0,X_0}\right],\\
R_{\varepsilon y}^{n,X_0} &= \frac{1}{2}\Delta t_n\bar{\varepsilon}_{f,n}^{x_0,X_0} + \frac{1}{2}\Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_{f,n+1}^{x_0,X_0}\right].
\end{aligned} \tag{4.5}$$

Define the perturbation errors of Scheme 3.1 as

$$\begin{aligned}
\varepsilon_y^{n,X_0} &= Y_{n,\varepsilon}^{X_0} - Y_n^{X_0}, \quad \varepsilon_{\bar{y}}^{n,X_0} = \bar{Y}_{n,\varepsilon}^{X_0} - \bar{Y}_n^{X_0},\\
\varepsilon_z^{n,X_0} &= Z_{n,\varepsilon}^{X_0} - Z_n^{X_0}, \quad \varepsilon_\gamma^{n,X_0} = \Gamma_{n,\varepsilon}^{X_0} - \Gamma_n^{X_0},
\end{aligned}$$

then by subtracting (3.13) and (3.16) from (4.1) and (4.4), respectively, we get the perturbation error equations

$$\begin{aligned}
\frac{1}{2}\Delta t_n\varepsilon_z^{n,X_0} = {} & \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_y^{n+1,X_0}\Delta\tilde{W}_n\right]\\
& + \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[\left(f_{n+1,\varepsilon}^{x_0,X_0} - f_{n+1}^{x_0,X_0}\right)\Delta\tilde{W}_n\right] + R_{\varepsilon z}^{n,X_0},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\frac{1}{2}\Delta t_n\varepsilon_\gamma^{n,X_0} = {} & \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_y^{n+1,X_0}\Delta\tilde{\mu}_n^*\right]\\
& + \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[\left(f_{n+1,\varepsilon}^{x_0,X_0} - f_{n+1}^{x_0,X_0}\right)\Delta\tilde{\mu}_n^*\right] + R_{\varepsilon\gamma}^{n,X_0},
\end{aligned} \tag{4.7}$$

$$\varepsilon_{\bar{y}}^{n,X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_y^{n+1,X_0}\right] + \Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0} - f_{n+1}^{x_0,X_0}\right] + \bar{R}_{\varepsilon y}^{n,X_0}, \tag{4.8}$$

$$\begin{aligned}
\varepsilon_y^{n,X_0} = {} & \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\varepsilon_y^{n+1,X_0}\right] + \frac{1}{2}\Delta t_n\left(\bar{f}_{n,\varepsilon}^{x_0,X_0} - \bar{f}_n^{x_0,X_0}\right)\\
& + \frac{1}{2}\Delta t_n\mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{n+1,\varepsilon}^{x_0,X_0} - f_{n+1}^{x_0,X_0}\right] + R_{\varepsilon y}^{n,X_0}.
\end{aligned} \tag{4.9}$$

Based on the above perturbation error equations, we first consider the stability of Scheme 3.1 for $X_0 = x_0$ in Theorem 4.1.

Theorem 4.1. *Suppose that f is uniformly Lipschitz continuous with a Lipschitz constant $L^{\dagger}$, and Assumption 2.1 holds. Then for sufficiently small time step Δt, we have (for $n=0,1,\ldots,N-1$)*

$$\begin{aligned}&\mathbb{E}\left[\left|\varepsilon_y^{n,x_0}\right|^2\right]+\Delta t\sum_{i=n}^{N-1}\mathbb{E}\left[\left|\varepsilon_z^{i,x_0}\right|^2+\left|\varepsilon_\gamma^{i,x_0}\right|^2\right]\\ \leq\,& C\left(\mathbb{E}\left[\left|\varepsilon_y^{N,x_0}\right|^2\right]+\Delta t\mathbb{E}\left[\left|\varepsilon_z^{N,x_0}\right|^2+\left|\varepsilon_z^{N,x_0}\right|^2\right]\right)\\ &+\frac{C}{\Delta t}\sum_{i=n}^{N-1}\mathbb{E}\left[(\Delta t)^2\left|\bar{R}_{\varepsilon y}^{i,x_0}\right|^2+\left|R_{\varepsilon y}^{i,x_0}\right|^2+\left|R_{\varepsilon z}^{i,x_0}\right|^2+\left|R_{\varepsilon\gamma}^{i,x_0}\right|^2\right],\end{aligned}\tag{4.10}$$

where $\Delta t=\max_{0\leq n\leq N-1}\Delta t_n$, and C is a positive constant depending on c_0 in (3.1), η, L and T.

Proof. For simplicity of notation, we let $X_n^{x_0}=X_n$ when $X_0=x_0$ and denote

$$\begin{aligned}&\left(\varepsilon_{\bar{y}}^n,\varepsilon_y^n,\varepsilon_z^n,\varepsilon_\gamma^n\right)=\left(\varepsilon_{\bar{y}}^{n,x_0},\varepsilon_y^{n,x_0},\varepsilon_z^{n,x_0},\varepsilon_\gamma^{n,x_0}\right),\\ &\left(\bar{R}_{\varepsilon y}^n,R_{\varepsilon y}^n,R_{\varepsilon z}^n,R_{\varepsilon\gamma}^n\right)=\left(\bar{R}_{\varepsilon y}^{n,x_0},R_{\varepsilon y}^{n,x_0},R_{\varepsilon z}^{n,x_0},R_{\varepsilon\gamma}^{n,x_0}\right),\\ &\left(f_n,\bar{f}_n\right)=\left(f_n^{x_0,x_0},\bar{f}_n^{x_0,x_0}\right),\quad\left(f_{n,\varepsilon},\bar{f}_{n,\varepsilon}\right)=\left(f_{n,\varepsilon}^{x_0,x_0},\bar{f}_{n,\varepsilon}^{x_0,x_0}\right).\end{aligned}$$

By the uniform Lipschitz continuity condition, we have

$$\left|f_{n,\varepsilon}-f_n\right|\leq L\left(\mathbb{E}\left[\left|\varepsilon_y^n\right|+\left|\varepsilon_z^n\right|+\left|\varepsilon_\gamma^n\right|\right]+\left|\varepsilon_y^n\right|+\left|\varepsilon_z^n\right|+\left|\varepsilon_\gamma^n\right|\right),\tag{4.11}$$

$$\left|\bar{f}_{n,\varepsilon}-\bar{f}_n\right|\leq L\left(\mathbb{E}\left[\left|\varepsilon_{\bar{y}}^n\right|+\left|\varepsilon_z^n\right|+\left|\varepsilon_\gamma^n\right|\right]+\left|\varepsilon_{\bar{y}}^n\right|+\left|\varepsilon_z^n\right|+\left|\varepsilon_\gamma^n\right|\right).\tag{4.12}$$

Then substituting (4.11) and (4.12) into (4.9), we deduce

$$\begin{aligned}\left|\varepsilon_y^n\right|\leq\,&\left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_y^{n+1}\right]\right|\\ &+\frac{1}{2}\Delta t_nL\Big(\mathbb{E}\left[\left|\varepsilon_{\bar{y}}^n\right|\right]+\left|\varepsilon_{\bar{y}}^n\right|+\mathbb{E}\left[\left|\varepsilon_z^n\right|+\left|\varepsilon_\gamma^n\right|\right]+\left|\varepsilon_z^n\right|+\left|\varepsilon_\gamma^n\right|\Big)\\ &+\frac{1}{2}\Delta t_nL\Big(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|+\left|\varepsilon_z^{n+1}\right|+\left|\varepsilon_\gamma^{n+1}\right|\right]\\ &\qquad+\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_y^{n+1}\right|+\left|\varepsilon_z^{n+1}\right|+\left|\varepsilon_\gamma^{n+1}\right|\right]\Big)+\left|R_{\varepsilon y}^n\right|.\end{aligned}\tag{4.13}$$

Similarly, by (4.8) and (4.11), we get

$$\begin{aligned}\left|\varepsilon_{\bar{y}}^n\right|\leq\,&\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_y^{n+1}\right|\right]\\ &+\Delta t_nL\Big(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|+\left|\varepsilon_z^{n+1}\right|+\left|\varepsilon_\gamma^{n+1}\right|\right]\\ &\qquad+\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_y^{n+1}\right|+\left|\varepsilon_z^{n+1}\right|+\left|\varepsilon_\gamma^{n+1}\right|\right]\Big)+\left|\bar{R}_{\varepsilon y}^n\right|.\end{aligned}$$

[†]The function f is uniformly Lipschitz continuous with the Lipschitz constant L, i.e., $|f(t,x_1',y_1',z_1',\gamma_1',x_1,y_1,z_1,\gamma_1)-f(t,x_2',y_2',z_2',\gamma_2',x_2,y_2,z_2,\gamma_2)|\leq L(|x_1'-x_2'|+|y_1'-y_2'|+|z_1'-z_2'|+|\gamma_1'-\gamma_2'|+|x_1-x_2|+|y_1-y_2|+|z_1-z_2|+|\gamma_1-\gamma_2|)$ for any $x_i',y_i',z_i',\gamma_i',x_i,y_i,z_i,\gamma_i\in\mathbb{R}$ with $i=1,2$.

Assume that $\Delta t L < 1$, then it is easy to obtain

$$\begin{aligned} \mathbb{E}\left[\left|\varepsilon_{\bar{y}}^{n}\right|\right] + \left|\varepsilon_{\bar{y}}^{n}\right| \leq{}& 4\mathbb{E}\left[\left|\varepsilon_{y}^{n+1}\right| + \left|\varepsilon_{z}^{n+1}\right| + \left|\varepsilon_{\gamma}^{n+1}\right|\right] + \mathbb{E}\left[\left|\bar{R}_{\varepsilon y}^{n}\right|\right] + \left|\bar{R}_{\varepsilon y}^{n}\right| \\ &+ 2\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_{y}^{n+1}\right| + \left|\varepsilon_{z}^{n+1}\right| + \left|\varepsilon_{\gamma}^{n+1}\right|\right]. \end{aligned} \tag{4.14}$$

By inserting (4.14) into (4.13), we have

$$\begin{aligned} \left|\varepsilon_{y}^{n}\right| \leq{}& \left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_{y}^{n+1}\right]\right| \\ &+ 3\Delta t_n L\Big(\mathbb{E}\left[\left|\varepsilon_{y}^{n+1}\right| + \left|\varepsilon_{z}^{n+1}\right| + \left|\varepsilon_{\gamma}^{n+1}\right|\right] + \mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_{y}^{n+1}\right| + \left|\varepsilon_{z}^{n+1}\right| + \left|\varepsilon_{\gamma}^{n+1}\right|\right] \\ &+ \mathbb{E}\left[\left|\varepsilon_{z}^{n}\right| + \left|\varepsilon_{\gamma}^{n}\right|\right] + \left|\varepsilon_{z}^{n}\right| + \left|\varepsilon_{\gamma}^{n}\right| + \mathbb{E}\left[\left|\bar{R}_{\varepsilon y}^{n}\right|\right] + \left|\bar{R}_{\varepsilon y}^{n}\right|\Big) + \left|R_{\varepsilon y}^{n}\right|. \end{aligned}$$

Apply the inequalities

$$(a+b)^2 \leq (1+\gamma\Delta t)a^2 + \left(1 + \frac{1}{\gamma\Delta t}\right)b^2$$

for some $\gamma > 0$ and $(\sum_{i=1}^{n} a_i)^2 \leq n\sum_{i=1}^{n} a_i^2$ with $n = 13$ to the above equation, and we get

$$\begin{aligned} \left|\varepsilon_{y}^{n}\right|^2 \leq{}& (1+\gamma\Delta t)\left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_{y}^{n+1}\right]\right|^2 + 117\left(1 + \frac{1}{\gamma\Delta t}\right) \\ &\times \Bigg((\Delta t_n L)^2\Big(\mathbb{E}\Big[\left|\varepsilon_{y}^{n+1}\right|^2 + \left|\varepsilon_{z}^{n+1}\right|^2 + \left|\varepsilon_{\gamma}^{n+1}\right|^2\Big] \\ &+ \mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_{y}^{n+1}\right|^2 + \left|\varepsilon_{z}^{n+1}\right|^2 + \left|\varepsilon_{\gamma}^{n+1}\right|^2\right] + \mathbb{E}\left[\left|\varepsilon_{z}^{n}\right|^2 + \left|\varepsilon_{\gamma}^{n}\right|^2\right] \\ &+ \left|\varepsilon_{z}^{n}\right|^2 + \left|\varepsilon_{\gamma}^{n}\right|^2 + \mathbb{E}\left[\left|\bar{R}_{\varepsilon y}^{n}\right|^2\right] + \left|\bar{R}_{\varepsilon y}^{n}\right|^2\Big) + \left|R_{\varepsilon y}^{n}\right|^2\Bigg). \end{aligned} \tag{4.15}$$

By using

$$(a+b)^2 \leq (1+\delta)a^2 + \left(1 + \frac{1}{\delta}\right)b^2$$

for some $\delta > 0$ and Hölder's inequality to (4.6), we deduce

$$\begin{aligned} \frac{1}{4}(\Delta t_n)^2\left|\varepsilon_{z}^{n}\right|^2 \leq{}& (1+\delta)\left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_{y}^{n+1}\Delta\tilde{W}_n\right]\right|^2 \\ &+ 2\left(1 + \frac{1}{\delta}\right)\left((\Delta t_n)^2\mathbb{E}_{t_n}^{X_n}\left[\left|f_{n+1,\varepsilon} - f_{n+1}\right|^2\right]\mathbb{E}_{t_n}^{X_n}\left[\left|\tilde{W}_n\right|^2\right] + \left|R_{\varepsilon z}^{n}\right|^2\right). \end{aligned} \tag{4.16}$$

From (4.16) and the following inequalities:

$$\begin{aligned} \mathbb{E}_{t_n}^{X_n}\left[\left|f_{n+1,\varepsilon} - f_{n+1}\right|^2\right] \leq{}& 6L^2\Big(\mathbb{E}\left[\left|\varepsilon_{y}^{n+1}\right|^2 + \left|\varepsilon_{z}^{n+1}\right|^2 + \left|\varepsilon_{\gamma}^{n+1}\right|^2\right] \\ &+ \mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_{y}^{n+1}\right|^2 + \left|\varepsilon_{z}^{n+1}\right|^2 + \left|\varepsilon_{\gamma}^{n+1}\right|^2\right]\Big), \\ \left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_{y}^{n+1}\Delta\tilde{W}_n\right]\right|^2 ={}& \left|\mathbb{E}_{t_n}^{X_n}\left[\left(\varepsilon_{y}^{n+1} - \mathbb{E}_{t_n}^{X_n}\left[\varepsilon_{y}^{n+1}\right]\right)\Delta\tilde{W}_n\right]\right|^2 \\ \leq{}& \Delta t_n\Big(\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_{y}^{n+1}\right|^2\right] - \left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_{y}^{n+1}\right]\right|^2\Big), \end{aligned}$$

we derive

$$\begin{aligned}\frac{1}{4}(\Delta t_n)^2\left|\varepsilon_z^n\right|^2 \le (1+\delta)\Delta t_n\Big(\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_y^{n+1}\right|^2\right]-\left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_y^{n+1}\right]\right|^2\Big)\\ +12\left(1+\frac{1}{\delta}\right)\Big(L^2(\Delta t_n)^3\Big(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2+\left|\varepsilon_z^{n+1}\right|^2+\left|\varepsilon_\gamma^{n+1}\right|^2\right]\\ +\mathbb{E}_{t_n}^{X_n}\left[\left|\varepsilon_y^{n+1}\right|^2+\left|\varepsilon_z^{n+1}\right|^2+\left|\varepsilon_\gamma^{n+1}\right|^2\right]\Big)+\left|R_{\varepsilon z}^n\right|^2\Big). \quad (4.17)\end{aligned}$$

Now we divide (4.17) by $2(1+\delta)(\Delta t_n)^2/\Delta t$ and take $\mathbb{E}[\cdot]$ on the derived equation to get

$$\begin{aligned}\frac{\Delta t}{8(1+\delta)}\mathbb{E}\left[\left|\varepsilon_z^n\right|^2\right] \le \frac{c_0}{2}\left(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2\right]-\mathbb{E}\left[\left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_y^{n+1}\right]\right|^2\right]\right)+\frac{6c_0^2}{\delta\Delta t}\mathbb{E}\left[\left|R_{\varepsilon z}^n\right|^2\right]\\ +\frac{12L^2(\Delta t)^2}{\delta}\left(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2+\left|\varepsilon_z^{n+1}\right|^2+\left|\varepsilon_\gamma^{n+1}\right|^2\right]\right). \quad (4.18)\end{aligned}$$

Similarly, we can deduce

$$\begin{aligned}\frac{\Delta t}{8\eta_0(1+\delta)}\mathbb{E}\left[\left|\varepsilon_\gamma^n\right|^2\right] \le \frac{c_0}{2}\left(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2\right]-\mathbb{E}\left[\left|\mathbb{E}_{t_n}^{X_n}\left[\varepsilon_y^{n+1}\right]\right|^2\right]\right)+\frac{6c_0^2}{\delta\eta_0\Delta t}\mathbb{E}\left[\left|R_{\varepsilon\gamma}^n\right|^2\right]\\ +\frac{12L^2(\Delta t)^2}{\delta}\left(\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2+\left|\varepsilon_z^{n+1}\right|^2+\left|\varepsilon_\gamma^{n+1}\right|^2\right]\right), \quad (4.19)\end{aligned}$$

where $\eta_0=\int_{\mathsf{E}}\eta^2(e)\lambda(de)$. Now by (4.15), (4.18) and (4.19), we deduce

$$\begin{aligned}&c_0\mathbb{E}\left[\left|\varepsilon_y^n\right|^2\right]+\frac{\Delta t}{8(1+\delta)}\mathbb{E}\left[\left|\varepsilon_z^n\right|^2\right]+\frac{\Delta t}{8\eta_0(1+\delta)}\mathbb{E}\left[\left|\varepsilon_\gamma^n\right|^2\right]\\ \le\ & c_0\left(1+\left(\gamma+234L^2\Delta t+\frac{234L^2}{\gamma}+\frac{24L^2\Delta t}{\delta c_0}\right)\Delta t\right)\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2\right]\\ &+\left(234c_0L^2\Delta t+\frac{234c_0L^2}{\gamma}+\frac{24L^2\Delta t}{\delta}\right)\Delta t\mathbb{E}\left[\left|\varepsilon_z^{n+1}\right|^2+\left|\varepsilon_\gamma^{n+1}\right|^2\right]\\ &+234c_0L^2\left(\Delta t+\frac{1}{\gamma}\right)\Delta t\mathbb{E}\left[\left|\varepsilon_z^n\right|^2+\left|\varepsilon_\gamma^n\right|^2\right]\\ &+234c_0L^2\left(\Delta t+\frac{1}{\gamma}\right)\frac{1}{\Delta t}\mathbb{E}\left[(\Delta t)^2\left|\bar{R}_{\varepsilon y}^n\right|^2+\left|R_{\varepsilon y}^n\right|^2\right]\\ &+\frac{6c_0^2}{\eta_0\delta\Delta t}\mathbb{E}\left[\eta_0\left|R_{\varepsilon z}^n\right|^2+\left|R_{\varepsilon\gamma}^n\right|^2\right],\end{aligned}$$

which can be written as

$$\begin{aligned}&c_0\mathbb{E}\left[\left|\varepsilon_y^n\right|^2\right]+C_1\Delta t\mathbb{E}\left[\left|\varepsilon_z^n\right|^2+\left|\varepsilon_\gamma^n\right|^2\right]\\ \le\ & c_0\left(1+C_2\Delta t\right)\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2\right]+C_3\Delta t\mathbb{E}\left[\left|\varepsilon_z^{n+1}\right|^2+\left|\varepsilon_\gamma^{n+1}\right|^2\right]\\ &+\frac{C_4}{\Delta t}\mathbb{E}\left[(\Delta t)^2\left|\bar{R}_{\varepsilon y}^n\right|^2+\left|R_{\varepsilon y}^n\right|^2\right]+\frac{C_5}{\Delta t}\mathbb{E}\left[\left|R_{\varepsilon z}^n\right|^2+\left|R_{\varepsilon\gamma}^n\right|^2\right], \quad (4.20)\end{aligned}$$

where

$$
\begin{aligned}
&C_1 = \frac{1+\eta_0}{8(1+\delta)\eta_0} - 234c_0L^2\left(\Delta t + \frac{1}{\gamma}\right),\\
&C_2 = \gamma + 234L^2\Delta t + \frac{234L^2}{\gamma} + \frac{24L^2\Delta t}{\delta c_0},\\
&C_3 = 234c_0L^2\Delta t + \frac{234c_0L^2}{\gamma} + \frac{24L^2\Delta t}{\delta},\\
&C_4 = 234c_0L^2\left(\Delta t + \frac{1}{\gamma}\right), \quad C_5 = \frac{6c_0^2(1+\eta_0)}{\delta\eta_0}.
\end{aligned}
$$

Taking $\delta = 1$ and choosing γ_0 to be large enough and Δt_0 small enough in (4.20), and letting $\gamma_0 \le \gamma \le 2\gamma_0$ and $0 < \Delta t \le \Delta t_0$, we get

$$
C_1 \le C, \quad C_2 \le C, \quad C_4 \le C, \quad C_5 \le C, \quad C_1 - C_3 > C^* > 0,
$$

where C and C^* are constants depending on c_0, η_0 and L. Then by (4.20), we obtain

$$
\begin{aligned}
&c_0\mathbb{E}\left[\left|\varepsilon_y^n\right|^2\right] + C_1\Delta t\mathbb{E}\left[\left|\varepsilon_z^n\right|^2 + \left|\varepsilon_\gamma^n\right|^2\right]\\
\le\ & (1+C\Delta t)\left(c_0\mathbb{E}\left[\left|\varepsilon_y^{n+1}\right|^2\right] + C_3\Delta t\mathbb{E}\left[\left|\varepsilon_z^{n+1}\right|^2 + \left|\varepsilon_\gamma^{n+1}\right|^2\right]\right)\\
&+ \frac{C}{\Delta t}\mathbb{E}\left[(\Delta t)^2\left|\bar{R}_{\varepsilon y}^n\right|^2 + \left|R_{\varepsilon y}^n\right|^2 + \left|R_{\varepsilon z}^n\right|^2 + \left|R_{\varepsilon\gamma}^n\right|^2\right],
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&c_0\mathbb{E}\left[\left|\varepsilon_y^n\right|^2\right] + C^*\Delta t\sum_{i=n}^{N-1}(1+C\Delta t)^{i-n}\mathbb{E}\left[\left|\varepsilon_z^i\right|^2 + \left|\varepsilon_\gamma^i\right|^2\right]\\
\le\ & (1+C\Delta t)^{N-n}\left(c_0\mathbb{E}\left[\left|\varepsilon_y^N\right|^2\right] + C_3\Delta t\mathbb{E}\left[\left|\varepsilon_z^N\right|^2 + \left|\varepsilon_\gamma^N\right|^2\right]\right)\\
&+ \sum_{i=n}^{N-1}(1+C\Delta t)^{i-n}\frac{C}{\Delta t}\mathbb{E}\left[(\Delta t)^2\left|\bar{R}_{\varepsilon y}^i\right|^2 + \left|R_{\varepsilon y}^i\right|^2 + \left|R_{\varepsilon z}^i\right|^2 + \left|R_{\varepsilon\gamma}^i\right|^2\right]\\
\le\ & C\left(\mathbb{E}\left[\left|\varepsilon_y^N\right|^2\right] + \Delta t\mathbb{E}\left[\left|\varepsilon_z^N\right|^2 + \left|\varepsilon_\gamma^N\right|^2\right]\right)\\
&+ \frac{C}{\Delta t}\sum_{i=n}^{N-1}\mathbb{E}\left[(\Delta t)^2\left|\bar{R}_{\varepsilon y}^i\right|^2 + \left|R_{\varepsilon y}^i\right|^2 + \left|R_{\varepsilon z}^i\right|^2 + \left|R_{\varepsilon\gamma}^i\right|^2\right],
\end{aligned}
$$

where the constant C depends on c_0, η, L and T. □

We give the stability results of Scheme 3.1 for $X_0 \ne x_0$ in the following theorem.

Theorem 4.2. *Under the conditions in Theorem 4.1, for sufficiently small time step Δt and $n = 0, 1, \ldots, N-1$, we have*

$$
\begin{aligned}
&\mathbb{E}\left[\left|\varepsilon_y^{n,X_0}\right|^2\right] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}\left[\left|\varepsilon_z^{i,X_0}\right|^2 + \left|\varepsilon_\gamma^{i,X_0}\right|^2\right] \\
\leq{}& C\left(\mathbb{E}\left[\left|\varepsilon_y^{N,x_0}\right|^2 + \left|\varepsilon_y^{N,X_0}\right|^2\right] + \Delta t \mathbb{E}\left[\left|\varepsilon_z^{N,x_0}\right|^2 + \left|\varepsilon_\gamma^{N,x_0}\right|^2 + \left|\varepsilon_z^{N,X_0}\right|^2 + \left|\varepsilon_\gamma^{N,X_0}\right|^2\right]\right) \\
&+ \frac{C}{\Delta t} \sum_{i=n}^{N-1} \mathbb{E}\left[(\Delta t)^2 \left|\bar{R}_{\varepsilon y}^{i,x_0}\right|^2 + \left|R_{\varepsilon y}^{i,x_0}\right|^2 + \left|R_{\varepsilon z}^{i,x_0}\right|^2 + \left|R_{\varepsilon \gamma}^{i,x_0}\right|^2\right] \\
&+ \frac{C}{\Delta t} \sum_{i=n}^{N-1} \mathbb{E}\left[(\Delta t)^2 \left|\bar{R}_{\varepsilon y}^{i,X_0}\right|^2 + \left|R_{\varepsilon y}^{i,X_0}\right|^2 + \left|R_{\varepsilon z}^{i,X_0}\right|^2 + \left|R_{\varepsilon \gamma}^{i,X_0}\right|^2\right],
\end{aligned}
$$

where C is a positive constant depending on c_0, η, L and T.

The above theorem follows from Theorem 4.1 by the similar arguments used in the proof of Theorem 4.1. So we omit it here.

Remark 4.1. From Theorem 4.2, we come to the conclusion that Scheme 3.1 is stable.

4.2. Error estimates

In this subsection, we will give the error estimates of Scheme 3.1 by applying the stability results in Theorem 4.2.

For notational simplicity, we let

$$
\begin{aligned}
&\left(Y_{t_n}^{X_0}, \bar{Y}_{t_n}^{X_0}, Z_{t_n}^{X_0}, \Gamma_{t_n}^{X_0}\right) \\
={}& \left(Y_{t_n}^{t_n, X_n^{X_0}}, \bar{Y}_{t_n}^{t_n, X_n^{X_0}}, Z_{t_n}^{t_n, X_n^{X_0}}, \Gamma_{t_n}^{t_n, X_n^{X_0}}\right) \\
={}& \left(Y_{t_n}\left(X_n^{X_0}\right), \bar{Y}_{t_n}\left(X_n^{X_0}\right), Z_{t_n}\left(X_n^{X_0}\right), \Gamma_{t_n}\left(X_n^{X_0}\right)\right),
\end{aligned}
$$

and define

$$
\begin{aligned}
\bar{f}_{t_n}^{x_0, X_0} &= \mathbb{E}\Big[f\left(t_n, \bar{\Theta}_{t_n}^{t_n, X_n^{x_0}}, \theta\right)\Big]\Big|_{\theta = \bar{\Theta}_{t_n}^{t_n, X_n^{X_0}}}, \\
f_{t_n}^{x_0, X_0} &= \mathbb{E}\Big[f\left(t_n, \Theta_{t_n}^{t_n, X_n^{x_0}}, \theta\right)\Big]\Big|_{\theta = \Theta_{t_n}^{t_n, X_n^{X_0}}}.
\end{aligned}
$$

Then the reference equations (3.8), (3.9), (3.11) and (3.12) can be rewritten as

$$
\frac{1}{2}\Delta t_n Z_{t_n}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{t_{n+1}}^{X_0} \Delta \tilde{W}_n\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{t_{n+1}}^{x_0, X_0} \Delta \tilde{W}_n\right] + R_z^{n, X_0} + \tilde{R}_z^{n, X_0}, \tag{4.21a}
$$

$$
\frac{1}{2}\Delta t_n \Gamma_{t_n}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{t_{n+1}}^{X_0} \Delta \tilde{\mu}_n^*\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{t_{n+1}}^{x_0, X_0} \Delta \tilde{\mu}_n^*\right] + R_\gamma^{n, X_0} + \tilde{R}_\gamma^{n, X_0}, \tag{4.21b}
$$

$$\bar{Y}_{t_n}^{X_0} = \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{t_{n+1}}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{t_{n+1}}^{x_0,X_0}\right] + \tilde{R}_{yr}^{n,X_0}, \tag{4.21c}$$

$$\begin{aligned} Y_{t_n}^{X_0} = &\mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{t_{n+1}}^{X_0}\right] + \frac{1}{2}\Delta t_n \bar{f}_{t_n}^{x_0,X_0} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{t_{n+1}}^{x_0,X_0}\right] \\ &+ R_y^{n,X_0} + \tilde{R}_y^{n,X_0}, \end{aligned} \tag{4.21d}$$

where $\tilde{R}_{yr}^{n,X_0}$, $\tilde{R}_y^{n,X_0}$, $\tilde{R}_z^{n,X_0}$ and $\tilde{R}_\gamma^{n,X_0}$ are defined as

$$\begin{aligned} \tilde{R}_z^{n,X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\big(Y_{t_{n+1}}^{t_n,X_n^{X_0}} - Y_{t_{n+1}}^{X_0}\big)\Delta\tilde{W}_n\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\big(f_{t_{n+1}}^{0,x_0,t_n,X_n} - f_{t_{n+1}}^{x_0,X_0}\big)\Delta\tilde{W}_n\right], \\ \tilde{R}_\gamma^{n,X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\big(Y_{t_{n+1}}^{t_n,X_n^{X_0}} - Y_{t_{n+1}}^{X_0}\big)\Delta\tilde{\mu}_n^*\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[\big(f_{t_{n+1}}^{0,x_0,t_n,X_n} - f_{t_{n+1}}^{x_0,X_0}\big)\Delta\tilde{\mu}_n^*\right], \\ \tilde{R}_{yr}^{n,X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{t_{n+1}}^{t_n,X_n^{X_0}} - Y_{t_{n+1}}^{X_0}\right] + \Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{t_{n+1}}^{0,x_0,t_n,X_n} - f_{t_{n+1}}^{x_0,X_0}\right], \\ \tilde{R}_y^{n,X_0} &= \mathbb{E}_{t_n}^{X_n^{X_0}}\left[Y_{t_{n+1}}^{t_n,X_n^{X_0}} - Y_{t_{n+1}}^{X_0}\right] + \frac{1}{2}\Delta t_n\left(\bar{f}_{t_n}^{0,x_0,t_n,X_n} - \bar{f}_{t_n}^{x_0,X_0}\right) \\ &\quad + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X_n^{X_0}}\left[f_{t_{n+1}}^{0,x_0,t_n,X_n} - f_{t_{n+1}}^{x_0,X_0}\right] \end{aligned} \tag{4.22}$$

with $\bar{f}_{t_n}^{0,x_0,t_n,X_n}$ and $f_{t_{n+1}}^{0,x_0,t_n,X_n}$ defined by

$$\begin{aligned} \bar{f}_{t_n}^{0,x_0,t_n,X_n} &= \mathbb{E}\Big[f\big(t_n, \bar{\Theta}_{t_n}^{0,x_0}, \theta\big)\Big]\Big|_{\theta=\bar{\Theta}_{t_n}^{t_n,X_n^{X_0}}}, \\ f_{t_{n+1}}^{0,x_0,t_n,X_n} &= \mathbb{E}\Big[f\big(t_{n+1}, \Theta_{t_{n+1}}^{0,x_0}, \theta\big)\Big]\Big|_{\theta=\Theta_{t_{n+1}}^{t_n,X_n^{X_0}}}. \end{aligned}$$

Since the equations in (4.21) have the same forms as the equations (4.1)-(4.4), we can take $(Y_{t_n}^{X_0}, Z_{t_n}^{X_0}, \Gamma_{t_n}^{X_0})$ as the solution of Scheme 3.1 with perturbations, i.e.,

$$\big(Y_{n,\varepsilon}^{X_0}, Z_{n,\varepsilon}^{X_0}, \Gamma_{n,\varepsilon}^{X_0}\big) = \big(Y_{t_n}^{X_0}, Z_{t_n}^{X_0}, \Gamma_{t_n}^{X_0}\big),$$

then the perturbation errors of Scheme 3.1 become its numerical errors, which are

$$e_y^{n,X_0} = Y_{t_n}^{X_0} - Y_n^{X_0}, \quad e_z^{n,X_0} = Z_{t_n}^{X_0} - Z_n^{X_0}, \quad e_\gamma^{n,X_0} = \Gamma_{t_n}^{X_0} - \Gamma_n^{X_0},$$

and the perturbation terms become

$$\big(\tilde{R}_{yr}^{n,X_0}, R_y^{n,X_0} + \tilde{R}_y^{n,X_0}, R_z^{n,X_0} + \tilde{R}_z^{n,X_0}, R_\gamma^{n,X_0} + \tilde{R}_\gamma^{n,X_0}\big).$$

Then by directly applying the stability results in Theorem 4.2, we deduce the error estimates of Scheme 3.1 in the following theorem.

Theorem 4.3. *Under the conditions in Theorem* 4.1*, for sufficiently small time step* Δt *and* $n = 0, 1, \ldots, N-1$*, we have*

$$\mathbb{E}\left[\big|e_y^{n,X_0}\big|^2\right] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}\left[\big|e_z^{i,X_0}\big|^2 + \big|e_\gamma^{i,X_0}\big|^2\right]$$

$$
\begin{aligned}
&\leq C\left(\mathbb{E}\left[\left|e_y^{N,x_0}\right|^2+\left|e_y^{N,X_0}\right|^2\right]+\Delta t\mathbb{E}\left[\left|e_z^{N,x_0}\right|^2+\left|e_\gamma^{N,x_0}\right|^2+\left|e_z^{N,X_0}\right|^2+\left|e_\gamma^{N,X_0}\right|^2\right]\right)\\
&\quad+\frac{C}{\Delta t}\sum_{i=n}^{N-1}\mathbb{E}\left[\left|R_y^{i,x_0}\right|^2+\left|R_z^{i,x_0}\right|^2+\left|R_\gamma^{i,x_0}\right|^2+\left|R_y^{i,X_0}\right|^2+\left|R_z^{i,X_0}\right|^2+\left|R_\gamma^{i,X_0}\right|^2\right]\\
&\quad+\frac{C}{\Delta t}\sum_{i=n}^{N-1}\mathbb{E}\left[(\Delta t)^2\left|\tilde{R}_{yr}^{i,x_0}\right|^2+\left|\tilde{R}_y^{i,x_0}\right|^2+\left|\tilde{R}_z^{i,x_0}\right|^2+\left|\tilde{R}_\gamma^{i,x_0}\right|^2\right]\\
&\quad+\frac{C}{\Delta t}\sum_{i=n}^{N-1}\mathbb{E}\left[(\Delta t)^2\left|\tilde{R}_{yr}^{i,X_0}\right|^2+\left|\tilde{R}_y^{i,X_0}\right|^2+\left|\tilde{R}_z^{i,X_0}\right|^2+\left|\tilde{R}_\gamma^{i,X_0}\right|^2\right],
\end{aligned}
$$

where C is a positive constant depending on c_0,η,L and T.

For the estimates of R_y^{n,X_0}, R_z^{n,X_0} and R_γ^{n,X_0} defined in the reference equations (3.8), (3.9) and (3.12), we have the following lemma.

Lemma 4.1. *Under Assumption 2.1, for $n=0,1,\ldots,N-1$, we have*

$$
\begin{aligned}
\mathbb{E}\left[\left|R_y^{n,X_0}\right|^2\right]&\leq C\left(1+\mathbb{E}\left[|x_0|^8+|X_0|^8\right]\right)(\Delta t)^6,\\
\mathbb{E}\left[\left|R_z^{n,X_0}\right|^2\right]&\leq C\left(1+\mathbb{E}\left[|x_0|^8+|X_0|^8\right]\right)(\Delta t)^6,\\
\mathbb{E}\left[\left|R_\gamma^{n,X_0}\right|^2\right]&\leq C\left(1+\mathbb{E}\left[|x_0|^8+|X_0|^8\right]\right)(\Delta t)^6,
\end{aligned}
$$

where C is a positive constant depending on η, T, and the upper bounds of the derivatives of the functions b,σ,c,f and Φ.

Proof. Based on the Feynman-Kac formulas in Lemma 2.1, by using Lemma 2.2, the Itô's formula (2.6) and the estimates of the solutions of MSDEJs in [10], the proof of Lemma 4.1 is standard. We omit it here. Interested readers can refer to [20, 22]. □

We also have the following estimates for $\tilde{R}_{yr}^{n,X_0}$, $\tilde{R}_y^{n,X_0}$, $\tilde{R}_z^{n,X_0}$ and $\tilde{R}_\gamma^{n,X_0}$ given in (4.22), which are generated by the Itô-Taylor scheme (3.2) for solving MSDEJs.

Lemma 4.2. *Assume that the conditions in Lemma 4.1 hold, then for $n=0,1,\ldots,N-1$, we have*

$$
\begin{aligned}
\mathbb{E}\left[\left|\tilde{R}_{yr}^{n,X_0}\right|^2\right]&\leq C\Big(1+\mathbb{E}\big[|x_0|^{4r_1}+|X_0|^{4r_1}\big]\Big)(\Delta t)^{2l+2},\\
\mathbb{E}\left[\left|\tilde{R}_y^{n,X_0}\right|^2\right]&\leq C\Big(1+\mathbb{E}\big[|x_0|^{4r_1}+|X_0|^{4r_1}\big]\Big)(\Delta t)^{2l+2},\\
\mathbb{E}\left[\left|\tilde{R}_z^{n,X_0}\right|^2\right]&\leq C\Big(1+\mathbb{E}\big[|x_0|^{4r_2}+|X_0|^{4r_2}\big]\Big)\big((\Delta t)^{2l+3}+(\Delta t)^{2\alpha_1+2}\big),\\
\mathbb{E}\left[\left|\tilde{R}_\gamma^{n,X_0}\right|^2\right]&\leq C\Big(1+\mathbb{E}\big[|x_0|^{4r_3}+|X_0|^{4r_3}\big]\Big)\big((\Delta t)^{2l+3}+(\Delta t)^{2\alpha_2+2}\big),
\end{aligned}
$$

where C is a positive constant independent of Δt and the values of α_1, α_2 and l depend on the specific Itô-Taylor schemes used to solve the forward MSDEJs.

Proof. Based on Feynman-Kac formulas in Lemma 2.1 and the estimates of numerical solutions of Itô-Taylor scheme (3.2) (see [21, Theorem 5.1]), Lemma 4.2 is a direct application of Proposition 3.1. □

Combining Lemmas 4.1-4.2 and Theorem 4.3, we obtain the error estimates of Scheme 3.1 in the following theorem.

Theorem 4.4. *Assume that the conditions in Lemma 4.2 and Theorem 4.3 hold, then for sufficiently small time step Δt and $n = 0, 1, \ldots, N-1$, we have*

$$\begin{aligned} &\mathbb{E}\Big[\big|e_y^{n,X_0}\big|^2\Big] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}\Big[\big|e_z^{i,X_0}\big|^2 + \big|e_\gamma^{i,X_0}\big|^2\Big] \\ &\le C\big((\Delta t)^{2\alpha_1} + (\Delta t)^{2\alpha_2} + (\Delta t)^{2l} + (\Delta t)^4\big), \end{aligned}$$

where C is a positive constant depending on $c_0, \eta, T, L, x_0, X_0$ and the upper bounds of the derivatives of b, σ, c, f and Φ.

Remark 4.2. From Remark 3.2 and the above theorem, we conclude that under certain regularity conditions, Scheme 3.1 is convergent with first order when the Euler scheme or the Milstein scheme are used, and second order when the weak order 2.0 Itô-Taylor scheme is used to solve MSDEJs.

5. Numerical experiments

To implement Scheme 3.1 into practice, we need to approximate the expectations $\mathbb{E}[\cdot]$ contained in the scheme (3.2) for solving MSDEJs and the conditional expectations $\mathbb{E}_{t_n}^x[\cdot]$ in Scheme 3.1 for solving MBSDEJs.

- For the approximations of $\mathbb{E}[\cdot]$ in the coefficients b, σ, c and f, we choose the Monte Carlo method.
- For the approximations of $\mathbb{E}_{t_n}^x[\cdot]$, we choose the Gaussian quadrature rules.

In this section, we first show how to approximate the expectations in the scheme (3.2) and the conditional expectations in Scheme 3.1. Then we present some numerical experiments to verify our theoretical results.

5.1. The approximations of the expectations

To apply Scheme 3.1, we first approximate the expectations contained in the scheme (3.2) by using the Monte-Carlo method. We shall take the following Euler scheme as an example to illustrate this procedure:

$$\begin{aligned} X_{n+1}^{X_0} = X_n^{X_0} &+ \mathbb{E}\big[\tilde{b}\big(t_n, X_n^{x_0}, x\big)\big]\big|_{x=X_n^{X_0}} \Delta t_n \\ &+ \mathbb{E}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big]\big|_{x=X_n^{X_0}} \Delta W_n \end{aligned}$$

$$+\sum_{i=1}^{\Delta N_n}\mathbb{E}\big[c\big(t_n,X_n^{x_0},x,e_i\big)\big]\big|_{x=X_n^{X_0}}.$$

Now we use the Monte-Carlo method to approximate the expectations in the above scheme to get

$$\begin{aligned}
\mathbb{E}\big[\tilde{b}\big(t_n,X_n^{x_0},x\big)\big] &= \frac{1}{M}\sum_{k=1}^{M}\tilde{b}\big(t_n,X_n^{x_0,k},x\big)+\mathcal{O}\left(\frac{1}{\sqrt{M}}\right),\\
\mathbb{E}\big[\sigma\big(t_n,X_n^{x_0},x\big)\big] &= \frac{1}{M}\sum_{k=1}^{M}\sigma\big(t_n,X_n^{x_0,k},x\big)+\mathcal{O}\left(\frac{1}{\sqrt{M}}\right),\\
\mathbb{E}\big[c\big(t_n,X_n^{x_0},x,e_i\big)\big] &= \frac{1}{M}\sum_{k=1}^{M}c\big(t_n,X_n^{x_0,k},x,e_i\big)+\mathcal{O}\left(\frac{1}{\sqrt{M}}\right),
\end{aligned}$$

where M is the sample times and $X_n^{x_0,k}$ is the numerical approximation solution at the time t_n obtained by the Euler scheme for MSDEJs at the k-th sampling. Denote by $\hat{\mathbb{E}}[\cdot]$ the approximated expectation obtained by the above Monte-Carlo method, i.e.,

$$\begin{aligned}
\hat{\mathbb{E}}\big[\tilde{b}\big(t_n,X_n^{x_0},x\big)\big] &= \frac{1}{M}\sum_{k=1}^{M}\tilde{b}\big(t_n,X_n^{x_0,k},x\big),\\
\hat{\mathbb{E}}\big[\sigma\big(t_n,X_n^{x_0},x\big)\big] &= \frac{1}{M}\sum_{k=1}^{M}\sigma\big(t_n,X_n^{x_0,k},x\big),\\
\hat{\mathbb{E}}\big[c\big(t_n,X_n^{x_0},x,e_i\big)\big] &= \frac{1}{M}\sum_{k=1}^{M}c\big(t_n,X_n^{x_0,k},x,e_i\big).
\end{aligned} \tag{5.1}$$

Then we can write the Euler scheme as

$$\begin{aligned}
X_{n+1}^{X_0} = X_n^{X_0} &+ \hat{\mathbb{E}}\big[\tilde{b}\big(t_n,X_n^{x_0},x\big)\big]\big|_{x=X_n^{X_0}}\Delta t_n\\
&+ \hat{\mathbb{E}}\big[\sigma\big(t_n,X_n^{x_0},x\big)\big]\big|_{x=X_n^{X_0}}\Delta W_n\\
&+ \sum_{i=1}^{\Delta N_n}\hat{\mathbb{E}}\big[c\big(t_n,X_n^{x_0},x,e_i\big)\big]\big|_{x=X_n^{X_0}}.
\end{aligned}$$

To be more specific, we solve the MSDEJs by the following two steps:

Step 1. Solve the MSDEJ with $X_0=x_0$ to obtain the values $\{X_n^{x_0,k}\}_{n=0}^{N}$

$$\begin{aligned}
X_{n+1}^{x_0,k} = X_n^{x_0,k} &+ \hat{\mathbb{E}}\big[\tilde{b}\big(t_n,X_n^{x_0},x\big)\big]\big|_{x=X_n^{x_0,k}}\Delta t_n\\
&+ \hat{\mathbb{E}}\big[\sigma\big(t_n,X_n^{x_0},x\big)\big]\big|_{x=X_n^{x_0,k}}\Delta W_n^k\\
&+ \sum_{i=1}^{\Delta N_n^k}\hat{\mathbb{E}}\big[c\big(t_n,X_n^{x_0},x,e_i^k\big)\big]\big|_{x=X_n^{x_0,k}}, \quad k=1,\dots,M,
\end{aligned}$$

where ΔW_n^k, ΔN_n^k and e_i^k are the k-th samples of ΔW_n, ΔN_n and e_i, respectively.

Step 2. Solve the MSDEJ with $X_0 \neq x_0$ to get the random variables $\{X_n^{X_0}\}_{n=0}^N$ after we get the values $\{X_n^{x_0,k}\}_{n=0}^N$

$$
\begin{aligned}
X_{n+1}^{X_0} = X_n^{X_0} &+ \hat{\mathbb{E}}\big[\tilde{b}\big(t_n, X_n^{x_0}, x\big)\big]\big|_{x=X_n^{X_0}} \Delta t_n \\
&+ \hat{\mathbb{E}}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big]\big|_{x=X_n^{X_0}} \Delta W_n \\
&+ \sum_{i=1}^{\Delta N_n} \hat{\mathbb{E}}\big[c\big(t_n, X_n^{x_0}, x, e_i\big)\big]\big|_{x=X_n^{X_0}}.
\end{aligned}
$$

5.2. The approximations of the conditional expectations

In this subsection, we shall show how to approximate the conditional expectations by using the Gaussian quadrature rules in detail. For simplicity, we write $X_n^{X_0} = X_n$, $Y_n^{X_0} = Y_n$ and show the approximation procedures of

$$
\mathbb{E}_{t_n}^x\big[Y_{n+1}\big], \quad \mathbb{E}_{t_n}^x\big[Y_{n+1}\Delta\tilde{W}_n\big], \quad \mathbb{E}_{t_n}^x\big[Y_{n+1}\Delta\tilde{\mu}_n^*\big]
$$

with the Euler scheme being used to solve MSDEJs. Moreover, we let

$$
\tilde{b}_n = \hat{\mathbb{E}}\big[\tilde{b}\big(t_n, X_n^{x_0}, x\big)\big], \quad \sigma_n = \hat{\mathbb{E}}\big[\sigma\big(t_n, X_n^{x_0}, x\big)\big], \quad c_{n,i} = \hat{\mathbb{E}}\big[c\big(t_n, X_n^{x_0}, x, e_i\big)\big],
$$

where $\hat{\mathbb{E}}[\cdot]$ is the approximated expectation defined as in (5.1) and $e_i \in \mathsf{E}$ is the i-th jump size for $i = 1, \ldots, \Delta N_n$ with $\Delta N_n = N_{t_{n+1}} - N_{t_n}$ the jump number occurring in $(t_n, t_{n+1}]$. Let $X_n = x$ and we have

$$
X_{n+1} = x + \tilde{b}_n \Delta t_n + \sigma_n \Delta W_n + \sum_{i=1}^{\Delta N_n} c_{n,i}.
$$

Suppose that the Lévy measure $\lambda(de)$ is in the form of

$$
\lambda(de) = \lambda\rho(e)de,
$$

where $\lambda = \lambda(\mathsf{E})$ is the intensity of μ and $\rho(e)$ is the probability density at e. Then

$$
\begin{aligned}
\mathbb{E}_{t_n}^x\big[Y_{n+1}\big] &= \mathbb{E}_{t_n}^x\big[Y_{n+1}\big(X_{n+1}\big)\big] \\
&= \mathbb{E}\left[Y_{n+1}\left(x + \tilde{b}_n \Delta t_n + \sigma_n \Delta W_n + \sum_{i=1}^{\Delta N_n} c_{n,i}\right)\right] \\
&= \mathbb{E}\left[\sum_{m=0}^{\infty} Y_{n+1}\left(x + \tilde{b}_n \Delta t_n + \sigma_n \Delta W_n + \sum_{i=1}^{m} c_{n,i}\right)\mathbb{I}_{\{\Delta N_n = m\}}\right] \\
&= \sum_{m=0}^{\infty} \mathbb{E}\left[Y_{n+1}\left(x + \tilde{b}_n \Delta t_n + \sigma_n \Delta W_n + \sum_{i=1}^{m} c_{n,i}\right)\right]\mathbb{P}\{\Delta N_n = m\} \\
&= \mathbb{E}_{t_n, M_y}^x\big[Y_{n+1}\big] + \mathcal{O}\left((\Delta t_n)^{M_y+1}\right),
\end{aligned} \tag{5.2}
$$

where M_y is the number of the truncated jumps, and

$$\mathbb{E}^x_{t_n,M_y}\big[Y_{n+1}\big] = \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!}\mathbb{E}\Big[Y_{n+1}\Big(x + \tilde{b}_n\Delta t_n + \sigma_n\Delta W_n + \sum_{i=1}^{m} c_{n,i}\Big)\Big]$$

is the approximation of $\mathbb{E}^x_{t_n}\big[Y_{n+1}\big]$. Since $\{e_1,\ldots,e_m\}$ are independent and identically distributed, we have

$$\begin{aligned}\mathbb{E}^x_{t_n,M_y}\big[Y_{n+1}\big] =& \sum_{m=0}^{M_y} \exp(-\lambda\Delta t_n)\frac{(\lambda\Delta t_n)^m}{m!}\\ &\times \int_{\mathbb{R}}\int_{\mathsf{E}}\cdots\int_{\mathsf{E}} Y_{n+1}\left(x + \tilde{b}_n\Delta t_n + \sigma_n\sqrt{\Delta t_n}s + \sum_{i=1}^{m} c_{n,i}\right)\\ &\times \frac{\exp(-s^2/2)}{\sqrt{2\pi}}\rho(e_1)\cdots\rho(e_m)\,dsde_1\cdots de_m,\end{aligned}$$

which can be approximated by appropriate Gaussian quadrature rules according to the probability density function $\rho(e)$.

The other two conditional expectations $\mathbb{E}^x_{t_n}[Y_{n+1}\Delta\tilde{W}_n]$ and $\mathbb{E}^x_{t_n}[Y_{n+1}\Delta\tilde{\mu}^*_n]$ can be approximated similarly, see more details in [27].

In our numerical tests, to keep Scheme 3.1 being second order convergent in time, by (5.2), we take $M_y = 2$. And we set the sample number in Monte Carlo method to be $M = 100000$ and the number of Gaussian quadrature points to be $L = 6$ such that the effect of the spatial approximation errors on the time discretization errors can be neglected.

5.3. Numerical examples

For simplicity, we take uniform partition in time with time step $\Delta t = T/N$ where N is a positive number. In all examples, we set the terminal time $T = 1.0$.

In the following tables, we denote by $|Y_0 - Y^0|$, $|Z_0 - Z^0|$ and $|\Gamma_0 - \Gamma^0|$ the errors between the exact solutions Y_t^{0,X_0}, Z_t^{0,X_0} and Γ_t^{0,X_0} of the MFBSDEJs (1.1) at $t = 0$ and the numerical solutions $Y_n^{X_0}$, $Z_n^{X_0}$ and $\Gamma_n^{X_0}$ of Scheme 3.1 at $n = 0$, respectively. The convergence rate (CR) with respect to Δt is obtained by using linear least square fitting of the errors.

Example 5.1. The considered MFBSDEJs model is

$$dX_t^{0,X_0} = bdt + \sigma dW_t + \int_{\mathsf{E}} c\tilde{\mu}(de,dt), \tag{5.3a}$$

$$\begin{aligned}-dY_t^{0,X_0} =& \bigg(Y_t^{0,X_0}\big(bX_t^{0,X_0} - 1\big) - \frac{\sigma}{2}Z_t^{0,X_0}\Big(\big(X_t^{0,X_0}\big)^2 - 1\Big)\\ &+\Gamma_t^{0,X_0} + \frac{1}{3}\mathbb{E}\left[\left(Y_t^{0,x_0} - \exp\left(t - \frac{1}{2}\big(X_t^{0,x_0}\big)^2\right)\right)^3\right]\bigg)\,dt\end{aligned}$$

$$- Z_t^{0,X_0} dW_t - \int_{\mathsf{E}} U_t^{0,X_0}(e)\tilde{\mu}(de,dt), \tag{5.3b}$$

$$Y_T^{0,X_0} = \exp\left(T - \frac{1}{2}(X_T^{0,X_0})^2\right), \tag{5.3c}$$

where the Lévy measure

$$\lambda(de) = \lambda\rho(e)de = \lambda\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{e^2}{2}\right)de$$

with $\lambda = \lambda(\mathsf{E})$ the intensity of μ. The analytic solutions Y_t^{0,X_0}, Z_t^{0,X_0} and Γ_t^{0,X_0} are

$$\begin{aligned}
Y_t^{0,X_0} &= \exp\left(t - \frac{1}{2}(X_t^{0,X_0})^2\right), \\
Z_t^{0,X_0} &= -\sigma X_t^{0,X_0}\exp\left(t - \frac{1}{2}(X_t^{0,X_0})^2\right), \\
\Gamma_t^{0,X_0} &= \lambda\left(\exp\left(t - \frac{1}{2}(X_t^{0,X_0}+c)^2\right) - \exp\left(t - \frac{1}{2}(X_t^{0,X_0})^2\right)\right).
\end{aligned}$$

Note that the solution of the MSDEJ in (5.3) is

$$X_t^{0,X_0} = X_0 + (b - c\lambda)t + \sigma W_t + cN_t,$$

and hence there is no error in solving the MSDEJ. Therefore, we can expect that Scheme 3.1 is second order accurate for solving the MFBSDEJs (5.3).

In our tests, we set $\lambda = 1.0$ and take the coefficients $b = 2.0$ and $\sigma = c = 1.0$, and solve (5.3) with different initial values of x_0 and X_0. All numerical results are listed in Tables 1 and 2.

All numerical results listed in Tables 1 and 2 show that Scheme 3.1 is stable and accurate for solving the decoupled MFBSDEJs (5.3) with different initial values of x_0 and X_0. Moreover, Scheme 3.1 is always convergent with second order when the MSDEJ in (5.3) has analytic solution. All numerical results are consistent with our theoretical conclusions.

Table 1: Errors and convergence rates of Scheme 3.1 with $x_0 = X_0$.

	$x_0 = X_0 = 0.0$			$x_0 = X_0 = -0.5$		
N	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert \Gamma_0 - \Gamma^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert \Gamma_0 - \Gamma^0\rvert$
16	3.344E-03	5.507e-03	6.382e-03	2.744E-03	1.396e-02	2.670e-03
32	8.452E-04	1.327e-03	1.634e-03	7.723E-04	3.683e-03	8.507e-04
64	2.092E-04	3.273e-04	4.142e-04	1.733E-04	9.010e-04	1.642e-04
128	4.827E-05	8.146e-05	1.043e-04	4.163E-05	2.231e-04	3.567e-05
256	7.731E-06	2.032e-05	2.619e-05	1.119E-05	5.623e-05	9.235e-06
CR	2.164	2.019	1.983	2.009	1.996	2.093

Table 2: Errors and convergence rates of Scheme 3.1 with $x_0 \neq X_0$.

	$x_0 = 0.0,\, X_0 = -1.0$			$x_0 = -0.5,\, X_0 = 0.5$		
N	$\lvert Y_0 - Y^0 \rvert$	$\lvert Z_0 - Z^0 \rvert$	$\lvert \Gamma_0 - \Gamma^0 \rvert$	$\lvert Y_0 - Y^0 \rvert$	$\lvert Z_0 - Z^0 \rvert$	$\lvert \Gamma_0 - \Gamma^0 \rvert$
16	5.962E-03	5.272e-03	1.202e-02	2.322E-03	6.616e-03	2.090e-03
32	1.549E-03	1.477e-03	3.102e-03	4.896E-04	1.463e-03	3.463e-04
64	4.060E-04	4.627e-04	8.026e-04	1.217E-04	3.673e-04	8.449e-05
128	1.064E-04	1.128e-04	2.017e-04	2.921E-05	9.063e-05	1.990e-05
256	3.117E-05	2.936e-05	5.077e-05	6.342E-06	2.185e-05	4.402e-06
CR	1.902	1.869	1.972	2.110	2.050	2.190

Example 5.2. Consider the following nonlinear MFBSDEJs:

$$\begin{aligned} dX_t^{0,X_0} &= \mathbb{E}\big[X_t^{0,x_0}\big]dt + \big(1 - x_0\exp(t) + \mathbb{E}\big[X_t^{0,x_0}\big]\big)dW_t + \int_{\mathsf{E}} e\tilde{\mu}(de,dt), \\ -dY_t^{0,X_0} &= \bigg(\frac{1}{2}Y_t^{0,X_0}\big(1 - x_0\exp(t) + \mathbb{E}\big[X_t^{0,x_0}\big]\big)^2 - Z_t^{0,X_0}\big(1 + \mathbb{E}\big[X_t^{0,x_0}\big]\big) \\ &\qquad\qquad - \Gamma_t^{0,X_0}\mathbb{E}\Big[\sin\big(2\big(t + X_t^{0,x_0}\big)\big) + \big(Y_t^{0,x_0}\big)^2\Big]\bigg)\,ds \\ &\quad - Z_t^{0,X_0}dW_t - \int_{\mathsf{E}} U_t^{0,X_0}(e)\tilde{\mu}(de,dt), \\ Y_T^{0,X_0} &= \sin\big(T + X_T^{0,X_0}\big) - \cos\big(T + X_T^{0,X_0}\big). \end{aligned} \tag{5.4}$$

In this example, we choose the Lévy measure

$$\lambda(de) = \lambda\rho(e)de = \frac{\lambda}{2\delta}\chi_{[-\delta,\delta]}(e)de$$

with the parameter $\delta > 0$. The analytic solution Y_t^{0,X_0}, Z_t^{0,X_0} and Γ_t^{0,X_0} are

$$\begin{aligned} Y_t^{0,X_0} &= \Big(\cos\big(t + X_t^{0,X_0}\big) + \sin\big(t + X_t^{0,X_0}\big)\Big)\Big(1 - x_0\exp(t) + \mathbb{E}\big[X_t^{0,x_0}\big]\Big), \\ \Gamma_t^{0,X_0} &= \frac{\lambda}{2\delta}\Big(\cos\big(t + X_t^{0,X_0} - \delta\big) - \cos\big(t + X_t^{0,X_0} + \delta\big) - 2\delta\sin\big(t + X_t^{0,X_0}\big)\Big) \\ &\quad - \frac{\lambda}{2\delta}\Big(\sin\big(t + X_t^{0,X_0} + \delta\big) - \sin\big(t + X_t^{0,X_0} - \delta\big) - 2\delta\cos\big(t + X_t^{0,X_0}\big)\Big). \end{aligned}$$

In our experiments, we take the intensity $\lambda = 2\delta$ and set $\delta = 0.5$, i.e., $\lambda = 1.0$. Then we implement Scheme 3.1 to solve the problem (5.4) with different initial values of x_0 and X_0. We test the Euler scheme (3.4), the Milstein scheme (3.5) and the weak order 2.0 Itô-Taylor scheme (3.6) for solving the MSDEJ in (5.4). These three schemes are denoted by Eul, Mil and W-2.0, respectively.

The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$ and $|\Gamma_0 - \Gamma^0|$, and their convergence rates are listed in the following Tables 3 and 4.

Table 3: Errors and convergence rates of Scheme 3.1 with $x_0 = X_0 = 0.5$.

		N = 8	N = 16	N = 32	N = 64	N = 128	CR
Eul	$\lvert Y_0 - Y^0\rvert$	4.495E-02	2.378E-02	1.202E-02	5.992E-03	3.099E-03	0.971
	$\lvert Z_0 - Z^0\rvert$	4.489E-02	2.179E-02	1.120E-02	5.638E-03	2.797E-03	0.996
	$\lvert \Gamma_0 - \Gamma^0\rvert$	4.481E-03	1.349E-03	5.482E-04	2.531E-04	1.293E-04	1.265
Mil	$\lvert Y_0 - Y^0\rvert$	4.495E-02	2.378E-02	1.202E-02	5.992E-03	3.099E-03	0.971
	$\lvert Z_0 - Z^0\rvert$	4.489E-02	2.179E-02	1.120E-02	5.638E-03	2.797E-03	0.996
	$\lvert \Gamma_0 - \Gamma^0\rvert$	4.481E-03	1.349E-03	5.482E-04	2.531E-04	1.293E-04	1.265
W-2.0	$\lvert Y_0 - Y^0\rvert$	1.269E-02	3.336E-03	8.703E-04	2.300E-04	2.282E-05	2.210
	$\lvert Z_0 - Z^0\rvert$	2.862E-02	6.475E-03	1.708E-03	4.699E-04	1.443E-04	1.905
	$\lvert \Gamma_0 - \Gamma^0\rvert$	1.813E-03	1.933E-04	1.274E-05	3.584E-06	7.142E-07	2.837

Table 4: Errors and convergence rates of Scheme 3.1 with $x_0 = 1.0$ and $X_0 = 0.0$.

		N = 8	N = 16	N = 32	N = 64	N = 128	CR
Eul	$\lvert Y_0 - Y^0\rvert$	6.082E-02	3.671E-02	1.990E-02	1.036E-02	5.316E-03	0.886
	$\lvert Z_0 - Z^0\rvert$	8.414E-02	3.696E-02	1.779E-02	8.753E-03	4.377E-03	1.061
	$\lvert \Gamma_0 - \Gamma^0\rvert$	5.696E-03	2.184E-03	9.713E-04	4.610E-04	2.273E-04	1.154
Mil	$\lvert Y_0 - Y^0\rvert$	6.082E-02	3.671E-02	1.990E-02	1.036E-02	5.316E-03	0.886
	$\lvert Z_0 - Z^0\rvert$	8.414E-02	3.696E-02	1.779E-02	8.753E-03	4.377E-03	1.061
	$\lvert \Gamma_0 - \Gamma^0\rvert$	5.696E-03	2.184E-03	9.713E-04	4.610E-04	2.273E-04	1.154
W-2.0	$\lvert Y_0 - Y^0\rvert$	3.219E-02	8.075E-03	2.084E-03	5.258E-04	8.621E-05	2.103
	$\lvert Z_0 - Z^0\rvert$	3.499E-02	8.082E-03	2.065E-03	5.887E-04	1.631E-04	1.927
	$\lvert \Gamma_0 - \Gamma^0\rvert$	1.395E-03	2.787E-04	5.883E-05	1.198E-05	4.858E-06	2.087

The numerical results in Tables 3 and 4 show that Scheme 3.1 is stable and accurate for solving the decoupled MFBSDEJs (5.4), and its accuracy depends on the methods used for solving the MSDEJ in (5.4). It is convergent with first order when the Euler scheme (3.4) and the Milstein scheme (3.5) are used to solve the MSDEJ, and is second order when the weak-order 2.0 Itô-Taylor scheme (3.6). All the numerical results admit a good match with our theoretical conclusions.

6. Conclusions

We proposed an explicit numerical scheme for solving decoupled MFBSDEJs. We rigorously analyzed the stability of the scheme and theoretically obtained its error estimates. Numerical results are presented to verify our theoretical conclusions, which show that the proposed scheme can be second order accurate when the weak order 2.0 Itô-Taylor scheme is used to solve the forward MSDEJ. In our future work, we shall focus on deep learning methods for solving high dimensional MFBSDEJs.

Acknowledgments

This research is partially supported by the NSF of China (Grant Nos. 12071261, 12001539, 11801320, 11831010, 12371398), by the National Key R&D Program of China (Grant No. 2018YFA0703900), by the NSF of Shandong Province (Grant No. ZR2023MA055), by the NSF of Hunan Province (Grant No. 2020JJ5647), and by the China Postdoctoral Science Foundation (Grant No. 2019TQ0073).

References

[1] A. Bensoussan, J. Frehse, and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Springer Briefs in Mathematics, 2013.

[2] M. Bossy and D. Talay, *A stochastic particle method for the McKean-Vlasov and the Burgers equation*, Math. Comput. 66 (1997), 157–192.

[3] R. Buckdahn, J. Li, and S. Peng, *Mean-field backward stochastic differential equations and related partial differential equations*, Stochastic Proc. Appl. 119 (2009), 3133–3154.

[4] R. Buckdahn, J. Li, S. Peng, and C. Rainer, *Mean-field stochastic differential equations and associated PDEs*, Ann. Probab. 45 (2017), 824–878.

[5] R. Carmona and F. Delarue, *Probabilistic analysis of mean-field games*, SIAM J. Control Optim. 51 (2012), 2705–2734.

[6] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/ CRC Press, 2004.

[7] W. E, J. Han, and Q. Li, *A mean-field optimal control formulation of deep learning*, Res. Math. Sci. 6 (2019), 1–41.

[8] P. Graber, *Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource*, Appl. Math. Optim. 74 (2016), 459–486.

[9] X. Guo, H. Pham, and X. Wei, *Itô's formula for flows of measures on semimartingales*, arXiv:2010.05288v4, 2022.

[10] T. Hao and J. Li, *Mean-field SDEs with jumps and nonlocal integral-PDEs*, Nonlinear Differ. Equ. Appl. 23 (2016), 1–51.

[11] P. Kotelenez, *A class of quasilinear stochastic partial differential equations of Mckean-Vlasov type with mass conservation*, Probab. Theory Relat. Fields 102 (1995), 159–188.

[12] J. Lasry and P. Lions, *Mean field games*, Japan J. Math. 2 (2007), 229–260.

[13] J. Li, *Stochastic maximum principle in the mean-field controls*, Automatica 48 (2012), 366–373.

[14] J. Li, *Mean-field forward and backward SDEs with jumps and associated nonlocal quasi-linear integral PDEs*, Stochastic Process. Appl. 128 (2018), 3118–3180.

[15] J. Li and H. Min, *Controlled mean-field backward stochastic differential equations with jumps involving the value function*, J. Syst. Sci. Comp. 29 (2016), 1–31.

[16] Y. Ni, X. Li, and J. Zhang, *Mean-field stochastic linear-quadratic optimal control with Markov jump parameters*, Systems Control Lett. 93 (2016), 69–76.

[17] E. Platen and N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance, Stochastic Modelling and Applied Probability*, Springer, 2010.

[18] Y. Sun, J. Yang, and W. Zhao, *Itô-Taylor schemes for solving mean-field stochastic differential equations*, Numer. Math. Theor. Meth. Appl. 10 (2017), 798–828.

[19] Y. SUN, J. YANG, W. ZHAO, AND T. ZHOU, *An explicit multistep scheme for mean-field forward-backward stochastic differential equations*, J. Comput. Math. 40 (2022), 519–543.

[20] Y. SUN AND W. ZHAO, *An explicit second-order numerical scheme for mean-field forward backward stochastic differential equations*, Numer. Algorithms 84 (2020), 253–283.

[21] Y. SUN AND W. ZHAO, *Numerical methods for mean-field stochastic differential equations with jumps*, Numer. Algorithms 88 (2021), 903–937.

[22] Y. SUN, W. ZHAO, AND T. ZHOU, *Explicit θ-scheme for solving mean-field backward stochastic differential equations*, SIAM J. Numer. Anal. 56 (2018), 2672–2697.

[23] M. TALBI, N. TOUZI, AND J. ZHANG, *Dynamic programming equation for the mean field optimal stopping problem*, arXiv:2103.05736v2, 2022.

[24] B. WANG AND J. ZHANG, *Mean field games for large-population multi-agent systems with Markov jump parameters*, SIAM J. Control Optim. 50 (2012), 2308–2334.

[25] G. WANG, C. ZHANG, AND W. ZHANG, *Stochastic maximum principle for mean-field type optimal control under partial information*, IEEE Trans. Autom. Control 59 (2014), 522–528.

[26] J. YONG, *A linear-quadratic optimal control problem for mean-field stochastic differential equations*, SIAM J. Control Optim. 51 (2011), 2809–2838.

[27] W. ZHAO, W. ZHANG, AND G. ZHANG, *Second-order numerical schemes for decoupled forward-backward stochastic differential equations with jumps*, J. Comput. Math. 35 (2017), 213–244.