Commun. Math. Res.
doi: 10.4208/cmr.2023-0025

Vol. **x**, No. x, pp. 1-29
x 2024

Analytic Regularity for a Singularly Perturbed Reaction-Convection-Diffusion Boundary Value Problem with Two Small Parameters

Irene Sykopetritou and Christos Xenophontos*

Department of Mathematics and Statistics, University of Cyprus, Po Box 20537, Nicosia 1678, Cyprus.

Received 21 May 2023; Accepted 2 November 2023

Abstract. We consider a second order, two-point, singularly perturbed boundary value problem, of reaction-convection-diffusion type with two small parameters, and we obtain analytic regularity results for its solution, under the assumption of analytic input data. First, we establish classical differentiability bounds that are explicit in the order of differentiation and the singular perturbation parameters. Next, for small values of these parameters we show that the solution can be decomposed into a smooth part, boundary layers at the two endpoints, and a negligible remainder. Derivative estimates are obtained for each component of the solution, which again are explicit in the differentiation order and the singular perturbation parameters.

AMS subject classifications: 34E05, 34E10

Key words: Singularly perturbed problem, reaction-convection-diffusion, boundary layers, analytic regularity.

1 Introduction

Singularly perturbed problems and the numerical approximation of their solution have been studied extensively over the last few decades (see, e.g., the books

*Corresponding author. *Email addresses:* `sykopetritou.eirini@ucy.ac.cy` (I. Sykopetritou), `xenophontos@ucy.ac.cy` (C. Xenophontos)

[8, 9, 12] and the references therein). As is well known, a main difficulty in these problems is the presence of boundary layers in the solution, which appear due to fact that the limiting problem (i.e. when the singular perturbation parameter(s) tend to 0), is of different order than the original one, and the ('extra') boundary conditions can only be satisfied if the solution varies rapidly in the vicinity of the boundary – hence the name boundary layers.

In most numerical methods, high order derivatives of the solution appear in the error estimates, hence one should have a clear picture of how these derivatives grow with respect to the singular perturbation parameter(s). For low order numerical methods, such as finite differences (FD) or the h version of the finite element method (FEM), derivatives up to order 3 are usually sufficient. For high order methods such as the hp version of the FEM, derivatives of arbitrary order are needed, thus knowing how these behave with respect to the singular perturbation parameter(s) as well as the differentiation order, is necessary. Usually problems of convection-diffusion or reaction-diffusion type are studied separately and several researchers have proposed and analyzed numerical schemes for the robust approximation of their solution (see, e.g., [12] and the references therein). When there are two singular perturbation parameters present in the differential equation, the problem becomes reaction-convection-diffusion and the relationship between the parameters determines the 'regime' we are in (as shown in Table 1 ahead). In [3], the numerical solution to this problem was addressed, using the h version of the FEM as well as appropriate finite differences (see also [1, 2, 4, 11, 13, 16, 17]). Our interest in is high order hp FEM, hence we require information on all derivatives of the solution. In the present article we obtain information about the analytic regularity of the solution, using the method of asymptotic expansions (see also [6]), thus providing the tools for an hp FEM for the approximation of such problems.

The rest of the paper is organized as follows: in Section 2 we present the model problem and the regularity of its solution in terms of classical differentiability. Section 3 contains the asymptotic expansion for the solution, under the assumption that the singular perturbation parameters are small enough. We consider all possible relationships between the singular perturbation parameters, and establish derivative bounds which are explicit in the differentiation order as well as the singular perturbation parameters. We also comment on the transition between the regimes, in the final subsection of Section 3. Finally, in Section 4 we summarize our conclusions.

With $I \subset \mathbb{R}$ an open, bounded interval with boundary ∂I and measure $|I|$, we will denote by $C^k(I)$ the space of continuous functions on I with continuous derivatives up to order k. We will use the usual Sobolev spaces $W^{k,m}(I)$ of functions on I with $0,1,2,\ldots,k$ generalized derivatives in $L^m(I)$, equipped with the norm and seminorm $\|\cdot\|_{k,m,I}$ and $|\cdot|_{k,m,I}$, respectively. When $m=2$, we will write $H^k(I)$ instead of $W^{k,2}(I)$, and for the norm and seminorm, we will write $\|\cdot\|_{k,I}$ and $|\cdot|_{k,I}$, respectively. The usual $L^2(I)$ inner product will be denoted by $\langle\cdot,\cdot\rangle_I$, with the subscript omitted when there is no confusion. We will also use the space

$$H_0^1(I)=\left\{u\in H^1(I): u|_{\partial I}=0\right\}.$$

The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted by $\|\cdot\|_{\infty,I}$. Finally, the notation "$a\lesssim b$" means "$a\leq Cb$" with C being a generic positive constant, independent of any discretization or singular perturbation parameters.

2 The model problem and its regularity

We consider the following model problem (cf. [10]): Find u such that

$$-\varepsilon_1 u''(x)+\varepsilon_2 b(x)u'(x)+c(x)u(x)=f(x), \quad x\in I=(0,1), \tag{2.1}$$
$$u(0)=u(1)=0, \tag{2.2}$$

where $0<\varepsilon_1,\, \varepsilon_2\leq 1$ are given parameters that can approach zero, and the functions b,c,f are given and sufficiently smooth. In particular, we assume that they are analytic functions satisfying, for some positive constants $\gamma_f,\gamma_c,\gamma_b$ independent of $\varepsilon_1,\varepsilon_2$,

$$\|f^{(n)}\|_{\infty,I}\lesssim n!\gamma_f^n, \quad \|c^{(n)}\|_{\infty,I}\lesssim n!\gamma_c^n, \quad \|b^{(n)}\|_{\infty,I}\lesssim n!\gamma_b^n, \quad \forall n=0,1,2,\ldots. \tag{2.3}$$

In addition, we assume that there exist constants β,γ,ρ, independent of $\varepsilon_1,\varepsilon_2$, such that for any $x\in\overline{I}=[0,1]$ there holds

$$b(x)\geq\beta>0, \quad c(x)\geq\gamma>0, \quad c(x)-\frac{\varepsilon_2}{2}b'(x)\geq\rho>0. \tag{2.4}$$

The solution to (2.1), (2.2) satisfies (see, e.g. [3])

$$\|u\|_{\infty,I}\lesssim 1. \tag{2.5}$$

We would like to obtain a similar estimate for u'. This is achieved in the following lemma.

Lemma 2.1. *Let u be the solution of* (2.1), (2.2) *and assume* (2.3), (2.4) *hold. Then*

$$\|u'\|_{\infty,I}\lesssim\varepsilon_1^{-1}.$$

Proof. The proof follows [7]. Let

$$A(x)=\frac{\varepsilon_2}{\varepsilon_1}\int_x^1 b(t)dt,$$

and note that $A(1)=0$ and $A'(x)=-(\varepsilon_2/\varepsilon_1)b(x)$. Multiplying (2.1) by $e^{A(x)}$ and integrating from x to 1 gives

$$-\varepsilon_1 u'(1)+\varepsilon_1 e^{A(x)}u'(x)+\int_x^1 e^{A(t)}c(t)u(t)dt=\int_x^1 e^{A(t)}f(t)dt.$$

Multiplying by $\varepsilon_1^{-1}e^{-A(x)}$ yields

$$u'(x)=e^{-A(x)}u'(1)-\frac{1}{\varepsilon_1}\int_x^1 e^{A(t)-A(x)}c(t)u(t)dt+\frac{1}{\varepsilon_1}\int_x^1 e^{A(t)-A(x)}f(t)dt. \tag{2.6}$$

Integrating from 0 to 1, we further get

$$0=u'(1)\int_0^1 e^{-A(x)}dx-\frac{1}{\varepsilon_1}\int_0^1\int_x^1 e^{A(t)-A(x)}\left[c(t)u(t)-f(t)\right]dtdx. \tag{2.7}$$

Since we wish to first estimate $u'(1)$, we need upper and lower bounds for the integral $\int_0^1 e^{-A(x)}dx$. From (2.4) we have

$$\int_0^1 e^{-A(x)}dx\leq\int_0^1 e^{-(\varepsilon_2/\varepsilon_1)\beta(1-x)}dx\leq\frac{\varepsilon_1}{\varepsilon_2\beta}. \tag{2.8}$$

Similarly,

$$\int_0^1 e^{-A(x)}dx\geq\int_0^1 e^{-(\varepsilon_2/\varepsilon_1)\|b\|_{\infty,I}(1-x)}dx=\frac{\varepsilon_1}{\varepsilon_2\|b\|_{\infty,I}}\left(1-e^{-(\varepsilon_2/\varepsilon_1)\|b\|_{\infty,I}}\right). \tag{2.9}$$

Also, to estimate the remaining terms in (2.7), we consider

$$\frac{1}{\varepsilon_1}\int_0^1\int_x^1 e^{A(t)-A(x)}dtdx=\frac{1}{\varepsilon_1}\int_0^1\int_x^1 e^{A'(\zeta)(t-x)}dtdx,$$

for some ζ between t and x. Hence,

$$\frac{1}{\varepsilon_1}\int_0^1\int_x^1 e^{A(t)-A(x)}dtdx\leq\frac{1}{\varepsilon_1}\int_0^1\int_x^1 e^{-(\varepsilon_2/\varepsilon_1)\beta(t-x)}dtdx\lesssim\frac{1}{\varepsilon_2}+\frac{\varepsilon_1}{\varepsilon_2^2}.$$

Using (2.7)-(2.9), we get

$$\begin{aligned}|u'(1)| &\lesssim \left(\int_0^1 e^{-A(x)}dx\right)^{-1}\left[\left(\|c\|_{\infty,I}\|u\|_{\infty,I}+\|f\|_{\infty,I}\right)\left(\frac{1}{\varepsilon_2}+\frac{\varepsilon_1}{\varepsilon_2^2}\right)\right]\\ &\lesssim \varepsilon_2\frac{\|b\|_{\infty,I}}{\varepsilon_1}\left(1-e^{-(\varepsilon_2/\varepsilon_1)\|b\|_{\infty,I}}\right)^{-1}\left(\frac{1}{\varepsilon_2}+\frac{\varepsilon_1}{\varepsilon_2^2}\right)\lesssim \varepsilon_1^{-1}.\end{aligned}$$

Inserting this bound in (2.6) gives

$$\begin{aligned}|u'(x)| &\lesssim \varepsilon_1^{-1}+\frac{1}{\varepsilon_1}\left(\|c\|_{\infty,I}\|u\|_{\infty,I}+\|f\|_{\infty,I}\right)\int_x^1 e^{A(t)-A(x)}dt\\ &\lesssim \varepsilon_1^{-1}+\frac{1}{\varepsilon_1}\left(\|c\|_{\infty,I}\|u\|_{\infty,I}+\|f\|_{\infty,I}\right)\int_x^1 e^{-(\varepsilon_2/\varepsilon_1)\beta(t-x)}dt\\ &\lesssim \varepsilon_1^{-1}+\frac{1}{\varepsilon_1}\left(\frac{\varepsilon_1}{\varepsilon_2\beta}\right)\lesssim \varepsilon_1^{-1},\end{aligned}$$

as desired. □

Using an inductive argument we are able to prove the following.

Theorem 2.1. *Let u be the solution of* (2.1), (2.2)*, and assume $\varepsilon_1 \le \varepsilon_2$. Then, there exist positive constants C,K independent of $\varepsilon_1,\varepsilon_2$ and u such that for $n=0,1,2,\dots$*

$$\|u^{(n)}\|_{\infty,I} \le CK^n \max\left\{n,\varepsilon_1^{-1}\right\}^n.$$

Proof. The proof is by induction on n and follows [6]. Eq. (2.5) and Lemma 2.1 give the result for $n=0,1$, so we assume it holds for $0\le\nu\le n+1$ and show that it holds for $n+2$. Differentiating (2.1) n times gives

$$\begin{aligned}-\varepsilon_1 u^{(n+2)} &= f^{(n)}-\varepsilon_2\left(bu'\right)^{(n)}-(cu)^{(n)}\\ &= f^{(n)}-\sum_{\nu=0}^{n}\binom{n}{\nu}\left(\varepsilon_2 b^{(\nu)}u^{(n+1-\nu)}+c^{(\nu)}u^{(n-\nu)}\right).\end{aligned}$$

By the induction hypothesis we have

$$\begin{aligned}\varepsilon_1\|u^{(n+2)}\|_{\infty,I} \le \|f^{(n)}\|_{\infty,I}+C\sum_{\nu=0}^{n}\binom{n}{\nu}&\Big[\varepsilon_2\gamma_b^\nu \nu! K^{n+1-\nu}\max\left\{n+1-\nu,\varepsilon_1^{-1}\right\}^{n+1-\nu}\\ &+\gamma_c^\nu \nu! K^{n-\nu}\max\left\{n-\nu,\varepsilon_1^{-1}\right\}^{n-\nu}\Big].\end{aligned}$$

Using the estimates below (which follow by standard considerations)

$$\begin{aligned}
&\binom{n}{\nu}\nu!\max\left\{n+1-\nu,\varepsilon_1^{-1}\right\}^{n+1-\nu}\le\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1},\\
&\binom{n}{\nu}\nu!\max\left\{n-\nu,\varepsilon_1^{-1}\right\}^{n-\nu}\le\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1},\\
&\|f^{(n)}\|_{\infty,I}\le C\gamma_f^n n!\le C\gamma_f^n\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1},
\end{aligned}$$

we obtain

$$\begin{aligned}
\varepsilon_1\|u^{(n+2)}\|_{\infty,I}\le{}& C\gamma_f^n\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1}\\
&+CK^{n+2}\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1}\sum_{\nu=0}^{n}\left[\frac{1}{K}\left(\frac{\gamma_b}{K}\right)^{\nu}+\frac{1}{K^2}\left(\frac{\gamma_c}{K}\right)^{\nu}\right]\\
\le{}& CK^{n+2}\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1}\left[\frac{1}{K^2}+\frac{1}{K}\frac{1}{(1-\gamma_b/K)}+\frac{1}{K^2}\frac{1}{(1-\gamma_c/K)}\right],
\end{aligned}$$

where we choose a constant $K>\max\{1,\gamma_f,\gamma_b,\gamma_c\}$ such that the expression in brackets above is bounded by 1. Thus

$$\varepsilon_1\|u^{(n+2)}\|_{\infty,I}\le CK^{n+2}\max\left\{n+1,\varepsilon_1^{-1}\right\}^{n+1}, \tag{2.10}$$

and dividing by ε_1 gives the desired result. □

Remark 2.1. The above result only treats the case $\varepsilon_1\le\varepsilon_2$, since if ε_2 is much smaller than ε_1, then we have a 'regular perturbation' of reaction-diffusion type. If one considers the limiting case $\varepsilon_2=0$, then one sees that there are two boundary layers, one at each endpoint, of width $\mathcal{O}(\varepsilon_1^{1/2})$. Hence, the result of Theorem 2.1 should read

$$\|u^{(n)}\|_{\infty,I}\le CK^n\max\left\{n,\varepsilon_1^{-1/2}\right\}^{n}.$$

More details arise if one studies the structure of the solution to (2.1), which depends on the roots of the characteristic equation associated with the differential operator. For this reason, we let $\lambda_0(x),\lambda_1(x)$ be the solutions of the characteristic equation and set

$$\mu_0=-\max_{x\in[0,1]}\lambda_0(x),\quad \mu_1=\min_{x\in[0,1]}\lambda_1(x), \tag{2.11}$$

or equivalently,

$$\mu_{0,1}=\min_{x\in[0,1]}\frac{\mp\varepsilon_2 b(x)+\sqrt{\varepsilon_2^2b^2(x)+4\varepsilon_1 c(x)}}{2\varepsilon_1}$$

with the minus sign associated with μ_0 and the plus sign with μ_1. The following hold true [13,16]:

$$\begin{cases} 1 \ll \mu_0 \le \mu_1, \quad \dfrac{\varepsilon_2}{\varepsilon_2+\varepsilon_1^{1/2}} \lesssim \varepsilon_2\mu_0 \lesssim 1, \quad \varepsilon_1^{1/2}\mu_0 \lesssim 1, \\ \max\left\{\mu_0^{-1},\varepsilon_1\mu_1\right\} \lesssim \varepsilon_1+\varepsilon_2^{1/2}, \qquad \varepsilon_2 \lesssim \varepsilon_1\mu_1, \\ \text{for } \varepsilon_2^2 \ge \varepsilon_1 : \varepsilon_1^{-1/2} \lesssim \mu_1 \lesssim \varepsilon_1^{-1}, \\ \text{for } \varepsilon_2^2 \le \varepsilon_1 : \varepsilon_1^{-1/2} \lesssim \mu_1 \lesssim \varepsilon_1^{-1/2}. \end{cases} \tag{2.12}$$

The values of μ_0, μ_1 determine the width of the boundary layers and since $t|\lambda_0(x)| < |\lambda_1(x)|$ the layer at $x=1$ is stronger than the layer at $x=0$. Essentially, there are three regimes [3]:

Table 1: Different regimes based on the relationship between ε_1 and ε_2 [3].

		μ_0	μ_1
convection-diffusion	$\varepsilon_1 \ll \varepsilon_2 = 1$	1	ε_1^{-1}
convection-reaction-diffusion	$\varepsilon_1 \ll \varepsilon_2^2 \ll 1$	ε_2^{-1}	$\varepsilon_2/\varepsilon_1$
reaction-diffusion	$1 \gg \varepsilon_1 \gg \varepsilon_2^2$	$\varepsilon_1^{-1/2}$	$\varepsilon_1^{-1/2}$

It was shown in [3] (see also [13]) that under the assumptions $b,c,f \in C^q(I)$ for some $q \ge 1$ and $\varepsilon_2 q\|b'\|_{\infty,I}/2 \lesssim (1-\ell)$ for some $\ell \in (0,1)$, the solution u to (2.1), (2.2) can be decomposed into a smooth part S, a boundary layer part at the left endpoint E_0 and a boundary layer part at the right endpoint E_1, viz.

$$u = S + E_0 + E_1 \tag{2.13}$$

with

$$\left|S^{(n)}(x)\right| \lesssim 1, \quad \left|E_0^{(n)}(x)\right| \lesssim \mu_0^n e^{-\ell\mu_0 x}, \quad \left|E_1^{(n)}(x)\right| \lesssim \mu_1^n e^{-\ell\mu_1(1-x)} \tag{2.14}$$

for all $x \in \overline{I}$ and for $n = 0,1,2,\ldots,q$. This regularity result is sufficient for proving convergence of a fixed order h FEM, but not for an hp FEM – a more refined regularity result is needed for the smooth part that shows how the derivatives grow, with respect to the differentiation order (cf. Eq. (3.19) ahead).

The above considerations suggest the following: If ε_1 is small compared to ε_2, then it is instructive to consider the limiting case $\varepsilon_1 = 0$. There is an exponential layer (of width $\mathcal{O}(\varepsilon_2)$) at the left endpoint. The homogeneous equation (with constant coefficients) suggests that the different regimes are $\varepsilon_1 \ll \varepsilon_2^2$, $\varepsilon_1 \approx \varepsilon_2^2$ and $\varepsilon_1 \gg \varepsilon_2^2$, as discussed below:

(1) In the regime $\varepsilon_1 \ll \varepsilon_2^2$, we have $\mu_0 = \mathcal{O}(\varepsilon_2^{-1})$ and $\mu_1 = \mathcal{O}(\varepsilon_2 \varepsilon_1^{-1})$. Hence, μ_1 is much larger than μ_0 and the boundary layer in the vicinity of $x=1$ is stronger. Consequently, there is a layer of width $\mathcal{O}(\varepsilon_2)$ at the left endpoint (the one that arises from the analysis of the case $\varepsilon_1 = 0$) and additionally, there is another layer at the right endpoint, of width $\mathcal{O}(\varepsilon_1/\varepsilon_2)$.

(2) In the regime $\varepsilon_1 \approx \varepsilon_2^2$ there are layers at both endpoints of width $\mathcal{O}(\varepsilon_2) = \mathcal{O}(\varepsilon_1^{1/2})$.

(3) In the regime $\varepsilon_2^2 \ll \varepsilon_1 \ll 1$, there are layers at both endpoints of width $\mathcal{O}(\varepsilon_1^{1/2})$.

The above information will be utilized in obtaining regularity estimates for the solution in all three regimes.

3 The asymptotic expansion

We elaborate on (1)-(3) above, and choose an appropriate asymptotic expansion for u, in what follows.

The proofs of each result in the subsequent sections are very similar, hence we will provide the details for Section 3.1 and omit certain proofs in Sections 3.2 and 3.3.

3.1 The regime $\varepsilon_1 \ll \varepsilon_2^2 \ll 1$

In this case we anticipate a layer of width $\mathcal{O}(\varepsilon_2)$ at the left endpoint and a layer of width $\mathcal{O}(\varepsilon_1/\varepsilon_2)$ at the right endpoint. To deal with this we define the stretched variables $\tilde{x} = x/\varepsilon_2$ and $\hat{x} = (1-x)\varepsilon_2/\varepsilon_1$, in order for the differentiation operator to produce the necessary powers of $\varepsilon_1, \varepsilon_2$, that yield a balanced (in $\varepsilon_1, \varepsilon_2$) equation.

Since we wish to stay along the lines of (2.13), we want the solution to be comprised of a smooth part (in the slow variable x), and two boundary layers (in the fast variables $\tilde{x}, \hat{x}$). Hence, we make the formal ansatz

$$u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon_2^i \left(\frac{\varepsilon_1}{\varepsilon_2^2} \right)^j \left(u_{i,j}(x) + \tilde{u}_{i,j}^{BL}(\tilde{x}) + \hat{u}_{i,j}^{BL}(\hat{x}) \right) \tag{3.1}$$

with $u_{i,j}, \tilde{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL}$ to be determined. Substituting (3.1) into (2.1), separating the slow and fast variables, and equating like powers of ε_1 and ε_2, we get (see [14] for the details)

$$\begin{cases} u_{0,0}(x)=\dfrac{f(x)}{c(x)}, & \\ u_{i,0}(x)=-\dfrac{b(x)}{c(x)}u'_{i-1,0}(x), & i\geq 1, \\ u_{0,j}(x)=u_{1,j}(x)=0, & j\geq 1, \\ u_{i,j}(x)=\dfrac{1}{c(x)}\left(u''_{i-2,j-1}(x)-b(x)u'_{i-1,j}(x)\right), & i\geq 2, \quad j\geq 1, \end{cases} \tag{3.2}$$

$$\begin{cases} \tilde{b}_0\left(\tilde{u}^{BL}_{0,0}\right)'+\tilde{c}_0\tilde{u}^{BL}_{0,0}=0, & \\ \tilde{b}_0\left(\tilde{u}^{BL}_{i,0}\right)'+\tilde{c}_0\tilde{u}^{BL}_{i,0}=-\displaystyle\sum_{k=1}^{i}\left(\tilde{b}_k\left(\tilde{u}^{BL}_{i-k,0}\right)'+\tilde{c}_k\tilde{u}^{BL}_{i-k,0}\right), & i\geq 1, \\ \tilde{b}_0\left(\tilde{u}^{BL}_{0,j}\right)'+\tilde{c}_0\tilde{u}^{BL}_{0,j}=\left(\tilde{u}^{BL}_{0,j-1}\right)'', & j\geq 1, \\ \tilde{b}_0\left(\tilde{u}^{BL}_{i,j}\right)'+\tilde{c}_0\tilde{u}^{BL}_{i,j}=\left(\tilde{u}^{BL}_{i,j-1}\right)''-\displaystyle\sum_{k=1}^{i}\left(\tilde{b}_k\left(\tilde{u}^{BL}_{i-k,j}\right)'+\tilde{c}_k\tilde{u}^{BL}_{i-k,j}\right), & i\geq 1, \quad j\geq 1, \end{cases} \tag{3.3}$$

$$\begin{cases} \left(\hat{u}^{BL}_{i,0}\right)''+\hat{b}_0\left(\hat{u}^{BL}_{i,0}\right)'=0, & i\geq 0, \\ \left(\hat{u}^{BL}_{0,j}\right)''+\hat{b}_0\left(\hat{u}^{BL}_{0,j}\right)'=\hat{c}_0\hat{u}^{BL}_{0,j-1}, & j\geq 1, \\ \left(\hat{u}^{BL}_{i,1}\right)''+\hat{b}_0\left(\hat{u}^{BL}_{i,1}\right)'=\hat{c}_0\hat{u}^{BL}_{i,0}-\hat{b}_1\left(\hat{u}^{BL}_{i-1,0}\right)', & i\geq 1, \\ \left(\hat{u}^{BL}_{1,j}\right)''+\hat{b}_0\left(\hat{u}^{BL}_{1,j}\right)'=\hat{c}_0\hat{u}^{BL}_{1,j-1}-\hat{b}_1\left(\hat{u}^{BL}_{0,j-1}\right)'+\hat{c}_1\hat{u}^{BL}_{0,j-2}, & j\geq 2, \\ \left(\hat{u}^{BL}_{i,j}\right)''+\hat{b}_0\left(\hat{u}^{BL}_{i,j}\right)'=\hat{c}_0\hat{u}^{BL}_{i,j-1}-\hat{b}_j\left(\hat{u}^{BL}_{i-j,0}\right)' & \\ \quad +\displaystyle\sum_{k=1}^{j-1}\left\{-\hat{b}_k\left(\hat{u}^{BL}_{i-k,j-k}\right)'+\hat{c}_k\hat{u}^{BL}_{i-k,j-k-1}\right\}, & i\geq 2, \quad j=2,\ldots,i, \\ \left(\hat{u}^{BL}_{i,j}\right)''+\hat{b}_0\left(\hat{u}^{BL}_{i,j}\right)'=\hat{c}_0\hat{u}^{BL}_{i,j-1} & \\ \quad +\displaystyle\sum_{k=1}^{i}\left\{-\hat{b}_k\left(\hat{u}^{BL}_{i-k,j-k}\right)'+\hat{c}_k\hat{u}^{BL}_{i-k,j-k-1}\right\}, & i\geq 2, \quad j>i, \end{cases} \tag{3.4}$$

where the notation $\tilde{b}_k(\tilde{x})=\tilde{x}^k b^{(k)}(0)/k!$, $\hat{b}_k(\hat{x})=(-1)^k\hat{x}^k b^{(k)}(1)/k!$ is used, and analogously for the other terms. (We also adopt the convention that empty sums are 0.) The BVPs (3.3)-(3.4) are supplemented with the following boundary conditions (in order for (2.2) to be satisfied) for all $i,j\geq 0$:

$$\tilde{u}_{i,j}^{BL}(0)=-u_{i,j}(0), \quad \lim_{\tilde{x}\to\infty}\tilde{u}_{i,j}^{BL}(\tilde{x})=0, \tag{3.5}$$

$$\hat{u}_{i,j}^{BL}(0)=-u_{i,j}(1), \quad \lim_{\hat{x}\to\infty}\hat{u}_{i,j}^{BL}(\hat{x})=0. \tag{3.6}$$

Next, we describe the regularity of the functions $u_{i,j},\tilde{u}_{i,j}^{BL},\hat{u}_{i,j}^{BL}$, defined by (3.2)-(3.5) above. We begin with $u_{i,j}$, and we have the following.

Lemma 3.1. *Let $u_{i,j}$ be defined by* (3.2) *and assume* (2.3) *holds. Then there exist positive constants C,K and a complex neighborhood G of $\overline{I}$ such that the complex extension of u (denoted again by u) satisfies*

$$|u_{i,j}(z)|\leq C\delta^{-i}K^i i^i, \quad \forall z\in G_\delta=\{z\in G: \mathrm{dist}(z,\partial G)>\delta\}.$$

Proof. The proof is by induction on i. The case $i=0$ holds trivially, so assume the result holds for i and establish it for $i+1$. Let $\kappa\in(0,1)$ and let $K>0$ be a constant so that $[2/K^2+1/K]\leq 1$. We have by (3.2), the induction hypothesis with $G_{(1-\kappa)\delta}\supset G_\delta$, and Cauchy's integral theorem for derivatives (we take as contour a circle of radius $\kappa\delta$ about $z_0\in G_\delta$),

$$\begin{aligned}\left|u_{i+1,j}(z)\right| &\leq C\left\{\left|u''_{i-1,j-1}(z)\right|+\left|u'_{i,j}(z)\right|\right\}\\ &\leq C\left\{\frac{2}{(\kappa\delta)^2}\big((1-\kappa)\delta\big)^{-i+1}K^{i-1}(i-1)^{i-1}+\frac{1}{(\kappa\delta)}\big((1-\kappa)\delta\big)^{-i}K^i i^i\right\}\\ &\leq C\delta^{-i-1}K^{i+1}(i+1)^{i+1}\left\{\frac{1}{K^2}\frac{1}{(i+1)^2}\frac{2}{\kappa^2(1-\kappa)^{i-1}}\left(\frac{i-1}{i+1}\right)^{i-1}\right.\\ &\qquad\left.+\frac{1}{K}\frac{1}{(i+1)}\frac{1}{\kappa(1-\kappa)^i}\left(\frac{i}{i+1}\right)^i\right\}.\end{aligned}$$

Choose $\kappa=1/(i+1)$. Then we get

$$\left|u_{i+1,j}(z)\right|\leq C\delta^{-i-1}K^{i+1}(i+1)^{i+1}\left[\frac{2}{K^2}+\frac{1}{K}\right],$$

so by the choice of K the expression in brackets is bounded by 1 and this completes the proof. ☐

Lemma 3.2. *Let $u_{i,j}$ be defined by* (3.2) *and assume* (2.3) *holds. Then there exist positive constants K_1,K_2 such that*

$$\left\|u_{i,j}^{(n)}\right\|_{\infty,I}\lesssim n!K_1^n i!K_2^i, \quad \forall n\in\mathbb{N}.$$

Proof. This follows immediately from Lemma 3.1 and Cauchy's integral theorem for derivatives

$$\left\| u_{i,j}^{(n)} \right\|_{\infty,I} \lesssim \frac{n!}{(n+1)^n} \delta^{-i} K^i i^i e^n \lesssim n! K_1^n i! K_2^i$$

with $K_1 = e, K_2 = K/\delta$. □

In order to treat the layer terms $\tilde{u}_{i,j}^{BL}$, $\hat{u}_{i,j}^{BL}$, we will develop some auxiliary results. The following one will be used in the proof of Lemma 3.4, and is an analog of [6, Lemma 7.3.6] (see also Proposition 3.1 ahead).

Lemma 3.3. *Let $\lambda,\gamma \in \mathbb{C}$ with $Re(\lambda) > 0, Re(\gamma) > 0$, and let $\alpha_1,\alpha_2 \in \mathbb{R}^+$. Suppose F is an entire function satisfying, for some $C_F > 0$, $i,j \in \mathbb{N}_0$,*

$$|F(z)| \leq C_F \gamma^{i+j} e^{-Re(\lambda z)} (\alpha_1 i + \alpha_2 j + |z|)^{\alpha_1 i + \alpha_2 j}, \quad \forall z \in \mathbb{C},$$

and let $v_0 \in \mathbb{C}$. Then, the solution $v:(0,\infty) \to \mathbb{C}$, of the problem

$$v' + \lambda v = F \quad on \ \ (0,\infty), \quad v(0) = v_0$$

can be extended to an entire function (denoted again by v), which satisfies

$$|v(z)| \leq \left[\frac{C_F}{|\lambda|} \frac{\gamma^{i+j}}{(\alpha_1 i + \alpha_2 j + 1)} (\alpha_1 i + \alpha_2 j + |z|)^{\alpha_1 i + \alpha_2 j + 1} + |v_0| \right] e^{-Re(\lambda z)}, \quad \forall z \in \mathbb{C}.$$

Proof. Using an integrating factor we find

$$v(z) = e^{-\lambda z} \left[v_0 + \int_0^{|z|} e^{\lambda s} F(s) ds \right],$$

from which we get

$$\begin{aligned} |v(z)| &\leq e^{-Re(\lambda z)} \left[|v_0| + \int_0^{|z|} \left| e^{Re(\lambda s)} F(s) \right| ds \right] \\ &\leq e^{-Re(\lambda z)} \left[|v_0| + \frac{C_F}{|\lambda|} \gamma^{i+j} \int_0^{|z|} \left| (\alpha_1 i + \alpha_2 j + |s|)^{\alpha_1 i + \alpha_2 j} \right| ds \right], \end{aligned}$$

where we used the assumption on F. The result follows. □

Lemma 3.4. *The functions $\tilde{u}_{i,j}^{BL}$ which satisfy* (3.3), (3.5), *are entire and there exist positive constants $C, \tilde{\gamma}$ such that*

$$\left| \left(\tilde{u}_{i,j}^{BL} \right)(z) \right| \leq C \frac{\tilde{\gamma}^{i+j}}{i!} (2i + j + |z|)^{2i+j} e^{-\beta Re(z)}, \quad z \in \mathbb{C}, \quad Re(z) > 0, \tag{3.7}$$

where $\beta = \tilde{c}_0 / \tilde{b}_0$.

Proof. We recall that

$$\tilde{b}_k(\tilde{x}) = \frac{\tilde{x}^k b^{(k)}(0)}{k!}, \quad \tilde{c}_k(\tilde{x}) = \frac{\tilde{x}^k c^{(k)}(0)}{k!}.$$

Consequently, there exist positive constants $C_{\tilde{b}}, \gamma_{\tilde{b}}, C_{\tilde{c}}, \gamma_{\tilde{c}}$, depending solely on b, c such that

$$\left|\tilde{b}_k(z)\right| \leq C_{\tilde{b}} \gamma_{\tilde{b}}^k |z|^k, \quad |\tilde{c}_k(z)| \leq C_{\tilde{c}} \gamma_{\tilde{c}}^k |z|^k. \tag{3.8}$$

Then, with K_2 the constant from Lemma 3.2, and $\gamma_{\tilde{b}}, \gamma_{\tilde{c}}$ given by (3.8), we choose $\tilde{\gamma} > \max\{K_2, \gamma_{\tilde{b}}, \gamma_{\tilde{c}}\}$ so that

$$\left[\frac{\gamma_{\tilde{b}}/\tilde{\gamma}}{1-\gamma_{\tilde{b}}/\tilde{\gamma}} + \frac{\gamma_{\tilde{c}}/\tilde{\gamma}}{1-\gamma_{\tilde{c}}/\tilde{\gamma}}\right] < 1. \tag{3.9}$$

Next, we note that from (3.3) we may calculate

$$\tilde{u}_{0,0}^{BL}(z) = -u_{0,0}(0) e^{-(\tilde{c}_0/\tilde{b}_0)z}.$$

Thus, using Lemma 3.2 to bound the term $|u_{0,0}(0)|$, we get

$$\left|\tilde{u}_{0,0}^{BL}(z)\right| \leq C e^{-|(\tilde{c}_0/\tilde{b}_0)z|} \leq C e^{-\beta Re(z)}, \quad \beta = \frac{\tilde{c}_0}{\tilde{b}_0},$$

thus the claim holds for $i, j = 0$. For $j = 0, i > 0$, we proceed with induction on i, while keeping j fixed at 0. We have shown the desired result for the case $i = 0$, so we assume it holds for $i > 0$ and we will establish it for $i+1$. The function $\tilde{u}_{i+1,0}^{BL}$ satisfies

$$\tilde{b}_0 \left(\tilde{u}_{i+1,0}^{BL}\right)' + \tilde{c}_0 \tilde{u}_{i+1,0}^{BL} = -\sum_{k=1}^{i+1} \left(\tilde{b}_k \left(\tilde{u}_{i+1-k,0}^{BL}\right)' + \tilde{c}_k \tilde{u}_{i+1-k,0}^{BL}\right) =: G_1,$$

as well as $\tilde{u}_{i+1,0}^{BL}(0) = -u_{i+1,0}(0)$. In order to use Lemma 3.3, we bound the right hand side above as follows:

$$\begin{aligned}
|G_1(z)| &\leq \sum_{k=1}^{i+1} \left[|\tilde{b}_k| \left| \left(\tilde{u}_{i+1-k,0}^{BL}\right)' \right| + |\tilde{c}_k| \left|\tilde{u}_{i+1-k,0}^{BL}\right|\right] \\
&\leq C \sum_{k=1}^{i+1} |z|^k \left[\left(\gamma_{\tilde{b}}^k + \gamma_{\tilde{c}}^k\right) \left|\tilde{u}_{i+1-k,0}^{BL}\right|\right],
\end{aligned}$$

where we used (3.8) and Cauchy's integral theorem for derivatives. The induction hypothesis yields

$$\begin{aligned}|G_1(z)| &\leq C\sum_{k=1}^{i+1} e^{-\beta Re(z)}|z|^k\left(\gamma_{\tilde{b}}^k+\gamma_{\tilde{c}}^k\right)\frac{\tilde{\gamma}^{i+1-k}}{(i+1-k)!}\left(2(i+1-k)+|z|\right)^{2(i+1-k)}\\ &\leq Ce^{-\beta Re(z)}\frac{\tilde{\gamma}^{i+1}}{i!}(2i+|z|)^{2i+1}\sum_{k=1}^{\infty}\left[\left(\frac{\gamma_{\tilde{b}}}{\tilde{\gamma}}\right)^k+\left(\frac{\gamma_{\tilde{c}}}{\tilde{\gamma}}\right)^k\right]\\ &\leq Ce^{-\beta Re(z)}\frac{\tilde{\gamma}^{i+1}}{i!}(2i+|z|)^{2i+1},\end{aligned}$$

since the geometric series converges to a quantity bounded by 1, by the choice of $\tilde{\gamma}$, see Eq. (3.9). Then, Lemma 3.3 yields

$$\left|\tilde{u}_{i+1,0}^{BL}(z)\right| \leq C\tilde{\gamma}^{i+1}e^{-\beta Re(z)}\left(\frac{(2(i+1)+|z|)^{2i+2}}{(i+1)!}+\frac{|u_{i+1,0}(0)|}{\tilde{\gamma}^{i+1}}\right).$$

Lemma 3.2, the choice of $\tilde{\gamma}$, and Stirling's formula, further give

$$\begin{aligned}\left|\tilde{u}_{i+1,0}^{BL}(z)\right| &\leq C\tilde{\gamma}^{i+1}e^{-\beta Re(z)}\left(\frac{(2(i+1)+|z|)^{2(i+1)}}{(i+1)!}+(i+1)^{i+1}\right)\\ &\leq C\tilde{\gamma}^{i+1}e^{-\beta Re(z)}\frac{(2(i+1)+|z|)^{2(i+1)}}{(i+1)!}\left[1+\frac{(i+1)^{2(i+1)}}{(2(i+1)+|z|)^{2(i+1)}}\right]\\ &\leq C\tilde{\gamma}^{i+1}e^{-\beta Re(z)}\frac{(2(i+1)+|z|)^{2(i+1)}}{(i+1)!}.\end{aligned}$$

This completes the induction on $i>0$ with $j=0$.

We next consider the case $i=0, j>0$. Assuming

$$\left|\tilde{u}_{0,j}^{BL}(z)\right| \leq C\tilde{\gamma}^j(j+|z|)^j e^{-\beta Re(z)},$$

we will establish it for $j+1$. The function $\tilde{u}_{0,j+1}^{BL}(z)$ satisfies for $j>0$,

$$\tilde{b}_0\left(\tilde{u}_{0,j+1}^{BL}\right)'+\tilde{c}_0\tilde{u}_{0,j+1}^{BL}=\left(\tilde{u}_{0,j}^{BL}\right)''.$$

By Cauchy's integral theorem for derivatives and the induction hypothesis, we have

$$\left|\left(\tilde{u}_{0,j}^{BL}(z)\right)''\right| \leq C\tilde{\gamma}^j(j+|z|)^j e^{-\beta Re(z)},$$

and by Lemmata 3.2, 3.3,

$$\begin{aligned}\left|\left(\tilde{u}^{BL}_{0,j+1}(z)\right)\right| &\leq C\tilde{\gamma}^j e^{-\beta Re(z)}\left\{\frac{(j+|z|)^{j+1}}{j+1}+\frac{|u_{0,j+1}|}{\tilde{\gamma}^j}\right\}\\ &\leq C\tilde{\gamma}^{j+1}e^{-\beta Re(z)}(j+1+|z|)^{j+1}\\ &\quad\times\left\{\frac{(j+|z|)^{j+1}}{\tilde{\gamma}(j+1)(j+1+|z|)^{j+1}}+\frac{\tilde{C}}{\tilde{\gamma}^{j+1}(j+1+|z|)^{j+1}}\right\}\\ &\leq C\tilde{\gamma}^{j+1}e^{-\beta Re(z)}(j+1+|z|)^{j+1}.\end{aligned}$$

This establishes the result for $i=0, j>0$.

We finally show the case $i,j>0$. We perform induction on $i>0$, while keeping j fixed (but arbitrary). We assume (3.7) holds for $i\geq 1$ and show it for $i+1$. We note that by (3.3), $\tilde{u}^{BL}_{i+1,j}$ satisfies

$$\tilde{b}_0\left(\tilde{u}^{BL}_{i+1,j}\right)'+\tilde{c}_0\tilde{u}^{BL}_{i+1,j}=\left(\tilde{u}^{BL}_{i+1,j-1}\right)''-\sum_{k=1}^{i+1}\left(\tilde{b}_k\left(\tilde{u}^{BL}_{i+1-k,j}\right)'+\tilde{c}_k\tilde{u}^{BL}_{i+1-k,j}\right)=:G_2,$$

as well as $\tilde{u}^{BL}_{i+1,j}(0)=-u_{i+1,j}(0)$. We bound G_2 using Cauchy's integral theorem for derivatives, (3.9), and the induction hypothesis

$$\begin{aligned}|G_2| &\leq \left|\left(\tilde{u}^{BL}_{i+1,j-1}\right)''\right|+\sum_{k=1}^{i+1}\left\{|\tilde{b}_k|\left|\left(\tilde{u}^{BL}_{i+1-k,j}\right)'\right|+|\tilde{c}_k|\left|\tilde{u}^{BL}_{i+1-k,j}\right|\right\}\\ &\leq Ce^{-\beta Re(z)}\tilde{\gamma}^{i+j}\frac{(2(i+1)+j-1+|z|)^{2(i+1)+j-1}}{(i+1)!}+\\ &\quad+Ce^{-\beta Re(z)}\tilde{\gamma}^{i+1+j}\sum_{k=1}^{i+1}\frac{(2(i+1-k)+j+|z|)^{2(i+1)-k+j}}{(i+1-k)!}\left[\left(\frac{\gamma_{\tilde{b}}}{\tilde{\gamma}}\right)^k+\left(\frac{\gamma_{\tilde{c}}}{\tilde{\gamma}}\right)^k\right]\\ &\leq Ce^{-\beta Re(z)}\tilde{\gamma}^{i+j+1}\frac{(2i+1+j+|z|)^{2i+1+j}}{i!},\end{aligned}$$

where we argued in a similar fashion as we did for G_1. Lemmata 3.2, 3.3 give

$$\begin{aligned}\left|\tilde{u}^{BL}_{i+1,j}\right| &\leq Ce^{-\beta Re(z)}\tilde{\gamma}^{i+1+j}\left\{\frac{(2(i+1)+j+|z|)^{2i+2+j}}{(i+1)!}+\frac{|u_{i+1,j}(0)|}{\tilde{\gamma}^{i+1+j}}\right\}\\ &\leq Ce^{-\beta Re(z)}\tilde{\gamma}^{i+1+j}\left\{\frac{(2(i+1)+j+|z|)^{2(i+1)+j}}{(i+1)!}+\frac{(i+1)^{i+1}}{\tilde{\gamma}^j}\right\}\\ &\leq Ce^{-\beta Re(z)}\tilde{\gamma}^{i+1+j}\frac{(2(i+1)+j+|z|)^{2(i+1)+j}}{(i+1)!}.\end{aligned}$$

This completes the proof. □

For the other layer term $\hat{u}_{i,j}^{BL}$, we have a similar result.

Lemma 3.5. *The functions* $\hat{u}_{i,j}^{BL}$ *which satisfy* (3.4), (3.5), *are entire and there exist positive constants* $C,\hat{\gamma}$ *such that*

$$\left|\left(\hat{u}_{i,j}^{BL}\right)(z)\right| \le C\frac{\hat{\gamma}^{i+j}}{j!}(i+2j+|z|)^{i+2j}e^{-\beta Re(z)}, \quad z\in\mathbb{C}, \quad Re(z)>0, \tag{3.10}$$

where $\beta=\hat{b}_0$.

Proof. The proof is very similar to that of Lemma 3.4, utilizing Lemmata 3.2, 3.3, and Cauchy's integral theorem for derivatives. The details appear in [14]. □

Using the previous two results, we obtain the following.

Lemma 3.6. *Let the functions* $\tilde{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL}$ *satisfy* (3.3), (3.4) *respectively. Then, there exist positive constants* $\tilde{C},\hat{C},\tilde{K},\hat{K},\tilde{\gamma},\hat{\gamma}$, *depending only on the data, such that* $\forall n\in\mathbb{N}$,

$$\left|\left(\tilde{u}_{i,j}^{BL}\right)^{(n)}(z)\right| \le \tilde{C}\tilde{K}^n\tilde{\gamma}^{i+j}\frac{(2i+j)^{2i+j}}{i!}e^{-\tilde{\beta}Re(z)}, \quad z\in\mathbb{C}, \quad Re(z)>0, \tag{3.11}$$

$$\left|\left(\hat{u}_{i,j}^{BL}\right)^{(n)}(z)\right| \le \hat{C}\hat{K}^n\hat{\gamma}^{i+j}\frac{(i+2j)^{i+2j}}{j!}e^{-\hat{\beta}Re(z)}, \quad z\in\mathbb{C}, \quad Re(z)>0, \tag{3.12}$$

where $\tilde{\beta}=\tilde{c}_0/\tilde{b}_0, \hat{\beta}=\hat{b}_0$.

Proof. We will prove (3.11), since (3.12) is similar. Cauchy's integral theorem for derivatives allows us to infer (3.11) from (3.7) as follows:

$$\begin{aligned}\left|\left(\tilde{u}_{i,j}^{BL}\right)^{(n)}(z)\right| &\le \tilde{C}e^{-\tilde{\beta}Re(z)}\frac{n!}{(n+1)^n}\tilde{\gamma}^{i+j}\frac{(2i+j+|z|)^{2i+j}}{i!}e^n \\ &\le \tilde{C}e^{-\tilde{\beta}Re(z)}\frac{n!}{(n+1)^n}\tilde{\gamma}^{i+j}\frac{(2i+j+n)^{2i+j}}{i!}e^n.\end{aligned}$$

Observing that

$$(2i+j+n)^{2i+j} = (2i+j)^{2i+j}\left(1+\frac{n}{2i+j}\right)^{2i+j} \le (2i+j)^{2i+j}e^n, \tag{3.13}$$

the result follows. □

We now define, for some $M\in\mathbb{N}$,

$$u_M(x)=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j u_{i,j}(x), \tag{3.14}$$

$$\tilde{u}_M^{BL}(\tilde{x})=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j \tilde{u}_{i,j}^{BL}(\tilde{x}), \tag{3.15}$$

$$\hat{u}_M^{BL}(\hat{x})=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j \hat{u}_{i,j}^{BL}(\hat{x}), \tag{3.16}$$

$$r_M=u-\left(u_M+\tilde{u}_M^{BL}+\hat{u}_M^{BL}\right) \tag{3.17}$$

and we have the following decomposition:

$$u=u_M+\tilde{u}_M^{BL}+\hat{u}_M^{BL}+r_M. \tag{3.18}$$

As the following theorem shows, the estimates on the smooth part u_M in (3.18), explicitly show the dependence on the differentiation order. Moreover, (3.19) shows that the smooth part is (real) analytic, hence a high order numerical method could produce exponential rates of convergence (see, e.g. [6]).

Theorem 3.1. *Assume* (2.3), (2.4) *hold, and that* $\varepsilon_1\ll\varepsilon_2^2$. *Then there exist positive constants* $K_1,K_2,\tilde{K},\hat{K},\tilde{\gamma},\hat{\gamma}$, *independent of* $\varepsilon_1,\varepsilon_2$, *such that the solution* u *of* (2.1)-(2.2) *can be decomposed as in* (3.18), *with*

$$\left\|u_M^{(n)}\right\|_{\infty,I}\lesssim n!K_1^n, \qquad \forall n\in\mathbb{N}_0, \tag{3.19}$$

$$\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(x)\right|\lesssim\tilde{K}^n\varepsilon_2^{-n}e^{-\beta x/\varepsilon_2}, \qquad \forall n\in\mathbb{N}_0, \tag{3.20}$$

$$\left|\left(\hat{u}_M^{BL}\right)^{(n)}(x)\right|\lesssim\hat{K}^n\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{-n}e^{-\beta(1-x)\varepsilon_2/\varepsilon_1}, \quad \forall n\in\mathbb{N}_0, \tag{3.21}$$

$$\|r_M\|_{\infty,\partial I}+\|r_M\|_{0,I}+\varepsilon_1^{1/2}\left\|r_M'\right\|_{0,I}\lesssim\max\left\{e^{-\beta\varepsilon_2/\varepsilon_1},e^{-\beta/\varepsilon_2}\right\}, \tag{3.22}$$

where M *is chosen so that*

$$\varepsilon_2e^2 4M\max\{1,K_2,\tilde{\gamma},\hat{\gamma}\}<1, \quad \frac{\varepsilon_1}{\varepsilon_2^2}e^2 4M\max\{1,\tilde{\gamma},\hat{\gamma}\}<1.$$

The constant β *is given by* $\beta=\min\{\tilde{c}_0/\tilde{b}_0,\hat{b}_0\}$.

Proof. We first show (3.19). From (3.14) and Lemma 3.2 we have

$$\begin{aligned}
\left\|u_M^{(n)}\right\|_{\infty,I} &\leq \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left\|u_{i,j}^{(n)}\right\|_{\infty,I} \\
&\lesssim \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j n!K_1^n i!K_2^i \\
&\lesssim n!K_1^n\left(\sum_{i=0}^{M}\varepsilon_2^i i^i K_2^i\right)\left(\sum_{j=0}^{M}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\right) \\
&\lesssim n!K_1^n\left(\sum_{i=0}^{\infty}(\varepsilon_2 M K_2)^i\right)\left(\sum_{j=0}^{\infty}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\right) \\
&\lesssim n!K_1^n,
\end{aligned}$$

since both sums are convergent geometric series due to the assumptions $\varepsilon_2 M K_2 < 1$ and $\varepsilon_1/\varepsilon_2^2 < 1$.

Next we show (3.20): By (3.15) and Lemma 3.6, we have

$$\begin{aligned}
\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(\tilde{x})\right| &\leq \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left|\left(\tilde{u}_{i,j}^{BL}\right)^{(n)}(\tilde{x})\right| \\
&\lesssim \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j \tilde{K}^n\tilde{\gamma}^{i+j}\frac{(2i+j)^{2i+j}}{i!}e^{-\beta\tilde{x}}.
\end{aligned}$$

Since $(2i+j)^{2i+j} \leq e^{2i}(2i)^{2i}e^j j^j$ (cf. (3.13)), we get

$$\begin{aligned}
\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(\tilde{x})\right| &\lesssim \tilde{K}^n e^{-\beta\tilde{x}}\left(\sum_{i=0}^{M}\frac{\tilde{\gamma}^i e^{2i}(2i)^{2i}\varepsilon_2^i}{i!}\right)\left(\sum_{j=0}^{M}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\tilde{\gamma}^j e^j j^j\right) \\
&\lesssim \tilde{K}^n e^{-\beta\tilde{x}}\left(\sum_{i=0}^{\infty}\left(\tilde{\gamma}e^2 4M\varepsilon_2\right)^i\right)\left(\sum_{j=0}^{\infty}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\tilde{\gamma}eM\right)^j\right) \\
&\lesssim \tilde{K}^n e^{-\beta\tilde{x}},
\end{aligned}$$

since both sums are convergent geometric series due to the assumptions $4\tilde{\gamma}e^2 M\varepsilon_2 < 1$, $(\varepsilon_1/\varepsilon_2^2)\tilde{\gamma}e^2 4M < 1$.

Similarly, we show (3.21): By (3.16) and Lemma 3.6

$$\begin{aligned}\left|\left(\hat{u}_M^{BL}\right)^{(n)}(\hat{x})\right| &\leq \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left|\left(\hat{u}_{i,j}^{BL}\right)^{(n)}(\hat{x})\right| \\ &\lesssim \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\hat{K}^n\hat{\gamma}^{i+j}\frac{(i+2j)^{i+2j}}{j!}e^{-\beta\hat{x}} \\ &\lesssim \hat{K}^n e^{-\beta\hat{x}}\left(\sum_{i=0}^{\infty}(\hat{\gamma}eM\varepsilon_2)^i\right)\left(\sum_{j=0}^{\infty}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\hat{\gamma}e^2 4M\right)^j\right) \\ &\lesssim \hat{K}^n e^{-\beta\hat{x}}.\end{aligned}$$

It remains to show (3.22). To this end, note that

$$\begin{aligned}r_M(0) &= u(0)-\left[\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left(u_{i,j}(0)+\tilde{u}_{i,j}^{BL}(0)+\hat{u}_{i,j}^{BL}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\right)\right] \\ &= -\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\hat{u}_{i,j}^{BL}\left(\frac{\varepsilon_2}{\varepsilon_1}\right).\end{aligned}$$

By (3.12),

$$\begin{aligned}|r_M(0)| &\leq \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left|\hat{u}_{i,j}^{BL}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\right| \\ &\lesssim \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\hat{\gamma}^{i+j}\frac{(i+2j)^{i+2j}}{j!}e^{-\beta\varepsilon_2/\varepsilon_1} \\ &\lesssim e^{-\beta\varepsilon_2/\varepsilon_1}\left(\sum_{i=0}^{\infty}(\hat{\gamma}Me\varepsilon_2)^i\right)\left(\sum_{j=0}^{\infty}\left(\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)\hat{\gamma}e^2 4M\right)^j\right) \\ &\lesssim e^{-\beta\varepsilon_2/\varepsilon_1}.\end{aligned}$$

Similarly,

$$\begin{aligned}|r_M(1)| &\leq \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left|\tilde{u}_{i,j}^{BL}\left(\frac{1}{\varepsilon_2}\right)\right| \\ &\lesssim \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\tilde{\gamma}^{i+j}\frac{(2i+j)^{2i+j}}{i!}e^{-\beta/\varepsilon_2}\end{aligned}$$

$$\begin{aligned}&\lesssim e^{-\beta/\varepsilon_2}\left(\sum_{i=0}^{\infty}(\tilde{\gamma}e^2\varepsilon_2 4M)^i\right)\left(\sum_{j=0}^{\infty}\left(\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)\tilde{\gamma}eM\right)^j\right)\\&\lesssim e^{-\beta/\varepsilon_2}.\end{aligned}$$

Combining the two results, we have

$$\|r_M\|_{\infty,\partial I}\lesssim\max\left\{e^{-\beta\varepsilon_2/\varepsilon_1},e^{-\beta/\varepsilon_2}\right\}.$$

Now, let

$$L:=-\varepsilon_1\frac{d^2}{dx^2}+\varepsilon_2 b\frac{d}{dx}+c\mathrm{Id}$$

with Id the identity operator, and consider

$$L(u-u_M)=f(x)-\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j Lu_{i,j}(x)$$

with $u_{i,j}$ satisfying (3.2). After some calculations, we find

$$L(u-u_M)=-\varepsilon_2^{M+1}\sum_{j=1}^{M}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j bu'_{M,j}.$$

Hence,

$$\begin{aligned}\|L(u-u_M)\|_{\infty,I}&\leq\varepsilon_2^{M+1}\sum_{j=1}^{M}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\|b\|_{\infty,I}\left\|u'_{M,j}\right\|_{\infty,I}\\&\lesssim\varepsilon_2^{M+1}\sum_{j=1}^{M}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left\|u'_{M,j}\right\|_{\infty,I}.\end{aligned}$$

Using Lemma 3.2, we further obtain

$$\|L(u-u_M)\|_{\infty,I}\lesssim\varepsilon_2^{M+1}M!K_2^M\sum_{j=1}^{M}\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\lesssim\varepsilon_2(\varepsilon_2 MK_2)^M,$$

since the finite sum can be bounded by a converging geometric series.

We also consider the operator L in the stretched variable $\tilde{x}$, and we find, after some calculations,

$$\begin{aligned}\tilde{L}\tilde{u}_M^{BL}&=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\tilde{L}\tilde{u}_{i,j}^{BL}\\&=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left\{-\varepsilon_1\varepsilon_2^{-2}\left(\tilde{u}_{i,j}^{BL}\right)''+\sum_{k=0}^{M}\left[\tilde{b}_k\left(\tilde{u}_{i,j}^{BL}\right)'+\tilde{c}_k\tilde{u}_{i,j}^{BL}\right]\right\}\end{aligned}$$

$$=\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M+1}\sum_{i=0}^{M}\varepsilon_2^i\left(\tilde{u}_{i,M}^{BL}\right)'',$$

where (3.3) was used. Hence, using (3.11), we have

$$\begin{aligned}\left\|\tilde{L}\tilde{u}_M^{BL}\right\|_{\infty,I}&\le\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M+1}\sum_{i=0}^{M}\varepsilon_2^i\left\|\left(\tilde{u}_{i,M}^{BL}\right)''\right\|_{\infty,I}\\&\lesssim\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M+1}\sum_{i=0}^{M}\varepsilon_2^i\tilde{\gamma}^{i+M}\frac{(2i+M)^{2i+M}}{i!}\\&\lesssim\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M+1}\sum_{i=0}^{M}\varepsilon_2^i\tilde{\gamma}^{i+M}e^{2i}(4i)^ie^MM^M\\&\lesssim\left(\frac{\varepsilon_1}{\varepsilon_2^2}\tilde{\gamma}eM\right)^{M+1}\sum_{i=0}^{M}\left(\varepsilon_2\tilde{\gamma}e^24M\right)^i\\&\lesssim\left(\frac{\varepsilon_1}{\varepsilon_2^2}\tilde{\gamma}eM\right)^{M+1}.\end{aligned}$$

Similarly, in the stretched variable $\hat{x}$ we have with the help of (3.4),

$$\begin{aligned}\hat{L}\hat{u}_M^{BL}&=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\hat{L}\hat{u}_{i,j}^{BL}\\&=\sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_2^i\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^j\left\{-\frac{\varepsilon_2^2}{\varepsilon_1}\left(\hat{u}_{i,j}^{BL}\right)''-\sum_{k=0}^{M}\left[\frac{\varepsilon_2^2}{\varepsilon_1}\hat{b}_k\left(\hat{u}_{i,j}^{BL}\right)'+\hat{c}_k\hat{u}_{i,j}^{BL}\right]\right\}\\&=\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M}\sum_{i=0}^{M}\varepsilon_2^i\left(\hat{u}_{i,M}^{BL}\right)'',\end{aligned}$$

and thus

$$\begin{aligned}\left\|\hat{L}\hat{u}_M^{BL}\right\|_{\infty,I}&\le\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M}\sum_{i=0}^{M}\varepsilon_2^i\left\|\left(\hat{u}_{i,M}^{BL}\right)''\right\|_{\infty,I}\\&\lesssim\left(\frac{\varepsilon_1}{\varepsilon_2^2}\right)^{M}\sum_{i=0}^{M}\varepsilon_2^i\hat{\gamma}^{i+M}\frac{(i+2M)^{i+2M}}{M!}\end{aligned}$$

$$\lesssim \left(\frac{\varepsilon_1}{\varepsilon_2^2}\hat{\gamma} e M\right)^M,$$

by following the exact same steps as above. Therefore,

$$\begin{aligned}\|Lr_M\|_{\infty,I} &= \left\|L\left(u-u_M-\tilde{u}_M^{BL}-\hat{u}_M^{BL}\right)\right\|_{\infty,I} \\ &\leq \|L(u-u_M)\|_{\infty,I} + \left\|\tilde{L}\tilde{u}_M^{BL}\right\|_{\infty,I} + \left\|\hat{L}\hat{u}_M^{BL}\right\|_{\infty,I} \\ &\lesssim \varepsilon_2(\varepsilon_2 M K_2)^M + \left(\frac{\varepsilon_1}{\varepsilon_2^2}\tilde{\gamma} e M\right)^{M+1} + \left(\frac{\varepsilon_1}{\varepsilon_2^2}\hat{\gamma} e M\right)^M.\end{aligned}$$

Under the assumptions of the theorem, we have shown that the remainder r_M has exponentially small values at the endpoints of I, and Lr_M is uniformly bounded by an arbitrarily small quantity on I. By stability (see, e.g., [3]) we have the desired result. □

The bounds of the previous theorem are of utmost importance in the design and proof of convergence (independently of $\varepsilon_1,\varepsilon_2$) of high order numerical methods, e.g. the *hp* finite element method (see, e.g. [15]). The bounds on the boundary layers tell us how to design the mesh for the approximation, so that the negative powers of $\varepsilon_1,\varepsilon_2$ are eliminated. The bounds on the smooth part, allow us to prove exponential convergence of the numerical method (see, e.g., [6]).

3.2 The regime $\varepsilon_1 \approx \varepsilon_2^2$

Now there are layers at both endpoints of width $\mathcal{O}(\varepsilon_2)$ and the BVP becomes reaction-diffusion like the one studied in [5]. So with $\tilde{x}=x/\varepsilon_2, \overline{x}=(1-x)/\varepsilon_2$, we make, analogously as in the previous case, the formal ansatz

$$u \sim \sum_{i=0}^{\infty} \varepsilon_2^i \left(u_i(x)+\tilde{u}_i^{BL}(\tilde{x})+\overline{u}_i^{BL}(\overline{x})\right) \tag{3.23}$$

with $u_i,\tilde{u}_i^{BL},\overline{u}_i^{BL}$ to be determined. Substituting (3.23) into (2.1), separating the slow and fast variables, and equating like powers of ε_1 ($\approx \varepsilon_2^2$) and ε_2 we get (see [14] for details)

$$\begin{aligned}&u_0(x)=\frac{f(x)}{c(x)}, \quad u_1(x)=-\frac{b(x)}{c(x)}u_0'(x), \\ &u_i(x)=\frac{1}{c(x)}\left(u_{i-2}''(x)-b(x)u_{i-1}'(x)\right), \quad i\geq 2,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&-\left(\tilde{u}_0^{BL}\right)''+\tilde{b}_0\left(\tilde{u}_0^{BL}\right)'+\tilde{c}_0\tilde{u}_0^{BL}=0,\\&-\left(\tilde{u}_i^{BL}\right)''+\tilde{b}_0\left(\tilde{u}_i^{BL}\right)'+\tilde{c}_0\tilde{u}_i^{BL}=-\sum_{k=1}^{i}\left(\tilde{b}_k\left(\tilde{u}_{i-k}^{BL}\right)'+\tilde{c}_k\tilde{u}_{i-k}^{BL}\right),\quad i\geq 1,\end{aligned}\tag{3.25}$$

$$\begin{aligned}&-\left(\overline{u}_i^{BL}\right)''+\bar{b}_0\left(\overline{u}_i^{BL}\right)'+\bar{c}_0\overline{u}_i^{BL}=0,\\&-\left(\overline{u}_i^{BL}\right)''+\bar{b}_0\left(\overline{u}_i^{BL}\right)'+\bar{c}_0\bar{u}_i^{BL}=\sum_{k=1}^{i}\left(\bar{b}_k\left(\bar{u}_{i-k}^{BL}\right)'-\bar{c}_k\bar{u}_{i-k}^{BL}\right),\quad i\geq 1,\end{aligned}\tag{3.26}$$

where the notation $\tilde{b}_k(\tilde{x})=\tilde{x}^k b^{(k)}(0)/k!$ etc., is used again. The above equations are supplemented with the following boundary conditions (in order to satisfy (2.2)):

$$\begin{aligned}&u_i(0)+\tilde{u}_i^{BL}(0)=0,\\&u_i(1)+\overline{u}_i^{BL}(0)=0,\\&\lim_{\tilde{x}\to\infty}\tilde{u}_i^{BL}(\tilde{x})=0,\quad \lim_{\overline{x}\to\infty}\overline{u}_i^{BL}(\overline{x})=0.\end{aligned}\tag{3.27}$$

We then define, for some $M\in\mathbb{N}$,

$$u_M(x)=\sum_{i=0}^{M}\varepsilon_2^i u_i(x),\quad \tilde{u}_M^{BL}(\tilde{x})=\sum_{i=0}^{M}\varepsilon_2^i\tilde{u}_i^{BL}(\tilde{x}),\quad \overline{u}_M^{BL}(\overline{x})=\sum_{i=0}^{M}\varepsilon_2^i\overline{u}_i^{BL}(\overline{x}),$$

as well as

$$u=u_M+\tilde{u}_M^{BL}+\overline{u}_M^{BL}+r_M.\tag{3.28}$$

We have the following theorem.

Theorem 3.2. *Assume* (2.3), (2.4) *hold, and that* $\varepsilon_1\approx\varepsilon_2^2$. *Then there exist positive constants* $K_1,K_2,\tilde{K},\overline{K},\delta$, *independent of* $\varepsilon_1,\varepsilon_2$, *such that the solution* u *of* (2.1)-(2.2) *can be decomposed as in* (3.28), *with*

$$\begin{aligned}&\left\|u_M^{(n)}\right\|_{\infty,I}\lesssim n!K_1^n, &&\forall n\in\mathbb{N}_0,\\&\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(x)\right|\lesssim\tilde{K}^n\varepsilon_2^{-n}e^{-\beta x/\varepsilon_2}, &&\forall n\in\mathbb{N}_0,\\&\left|\left(\overline{u}_M^{BL}\right)^{(n)}(x)\right|\lesssim\overline{K}^n\varepsilon_2^{-n}e^{-\beta(1-x)/\varepsilon_2}, &&\forall n\in\mathbb{N}_0,\\&\|r_M\|_{\infty,\partial I}+\|r_M\|_{0,I}+\varepsilon_2\left\|r_M'\right\|_{0,I}\lesssim e^{-\delta/\varepsilon_2},\end{aligned}$$

where M *is chosen so that* $\varepsilon_2K_2M<1$, *and* $\beta=\min\{\tilde{c}_0/\tilde{b}_0,\overline{c}_0/\overline{b}_0\}$.

Proof. When $\varepsilon_1 \approx \varepsilon_2^2$, the BVP (2.1)-(2.2) becomes

$$\begin{aligned} &-u''(x)+\varepsilon_2^{-1}b(x)u'(x)+\varepsilon_2^{-2}c(x)u(x)=\varepsilon_2^{-2}f(x), \quad x\in I=(0,1),\\ &u(0)=u(1)=0. \end{aligned}$$

Multiplying the differential equation above by $e^{-\int_0^x \varepsilon_2^{-1}b(t)dt}$, gives

$$-\left(u'(x)e^{-\int_0^x \varepsilon_2 b(t)dt}\right)'+e^{-\int_0^x \varepsilon_2^{-1}b(t)dt}c(x)u(x)=e^{-\int_0^x \varepsilon_2^{-1}b(t)dt}f(x),$$

or equivalently,

$$\begin{aligned} &-\varepsilon_2^2 v''(x)+c(x)v(x)=F(x), \quad x\in I=(0,1),\\ &v(0)=v(1)=0, \end{aligned}$$

where

$$v(x)=e^{-\int_0^x \varepsilon_2^{-1}b(t)dt}u(x), \quad F(x)=e^{-\int_0^x \varepsilon_2^{-1}b(t)dt}f(x).$$

The above BVP is in the form considered in [5], with $c(x)$, $F(x)$ analytic – the analyticity of $F(x)$ follows from the analyticity of b and f. The desired bounds follow from the results in [5] and the fact that $|u^{(n)}|<|v^{(n)}|$. □

3.3 The regime $\varepsilon_2^2 \ll \varepsilon_1 \ll 1$

We anticipate layers at both endpoints of width $\mathcal{O}(\sqrt{\varepsilon_1})$. So we define the stretched variables $\check{x}=x/\sqrt{\varepsilon_1}$ and $\grave{x}=(1-x)/\sqrt{\varepsilon_1}$ and make the formal ansatz, analogous to the previous cases,

$$u\sim\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varepsilon_1^{i/2}\left(\frac{\varepsilon_2}{\sqrt{\varepsilon_1}}\right)^j\left(u_{i,j}(x)+\check{u}_{i,j}^{BL}(\check{x})+\grave{u}_{i,j}^{BL}(\grave{x})\right) \tag{3.29}$$

with $u_{i,j}$, $\check{u}_{i,j}^{BL}$, $\grave{u}_{i,j}^{BL}$ to be determined. Substituting (3.29) into (2.1), separating the slow and fast variables, and equating like powers of ε_1 and ε_2 we get (see [14] for the details)

$$\begin{cases} u_{0,0}=\dfrac{f(x)}{c(x)}, \quad u_{1,0}(x)=u_{0,j}(x)=0, & j\geq 1,\\ u_{i,0}(x)=\dfrac{1}{c(x)}u''_{i-2,0}(x), & i\geq 2,\\ u_{2i+1,0}(x)=0, & i\geq 1,\\ u_{1,1}(x)=-\dfrac{b(x)}{c(x)}u'_{0,0}(x), \quad u_{1,j}(x)=0, & j\geq 2,\\ u_{i,j}(x)=\dfrac{1}{c(x)}\left(u''_{i-2,j}(x)-b(x)u'_{i-1,j-1}(x)\right), & i\geq 2, \quad j\geq 1, \end{cases} \tag{3.30}$$

$$\begin{cases} -\left(\breve{u}_{0,0}^{BL}\right)'' + \breve{c}_0 \breve{u}_{0,0}^{BL} = 0, \\ -\left(\breve{u}_{i,0}^{BL}\right)'' + \breve{c}_0 \breve{u}_{i,0}^{BL} = -\sum_{k=i}^{i} \breve{c}_k \breve{u}_{i-k,0}^{BL}, & i \geq 1, \\ -\left(\breve{u}_{0,j}^{BL}\right)'' + \breve{c}_0 \breve{u}_{0,j}^{BL} = -\breve{b}_0 \left(\breve{u}_{0,j-1}^{BL}\right)', & j \geq 1, \\ -\left(\breve{u}_{i,j}^{BL}\right)'' + \breve{c}_0 \breve{u}_{i,j}^{BL} = -\breve{b}_0 \left(\breve{u}_{i,j-1}^{BL}\right)' \\ \quad -\sum_{k=1}^{i} \left\{ \breve{b}_k \left(\breve{u}_{i-k,j-1}^{BL}\right)' + \breve{c}_k \breve{u}_{i-k,j}^{BL} \right\}, & i \geq 1, \quad j \geq 1, \end{cases} \tag{3.31}$$

$$\begin{cases} -\left(\grave{u}_{0,0}^{BL}\right)'' + \grave{c}_0 \grave{u}_{0,0}^{BL} = 0, \\ -\left(\grave{u}_{i,0}^{BL}\right)'' + \grave{c}_0 \grave{u}_{i,0}^{BL} = -\sum_{k=1}^{i} \grave{c}_k \grave{u}_{i-k,0}^{BL}, & i \geq 1, \\ -\left(\grave{u}_{0,j}^{BL}\right)'' + \grave{c}_0 \grave{u}_{0,j}^{BL} = \grave{b}_0 \grave{u}_{0,j-1}^{BL}, & j \geq 1, \\ -\left(\grave{u}_{i,j}^{BL}\right)'' + \grave{c}_0 \grave{u}_{i,j}^{BL} = \left(\grave{b}_0 \grave{u}_{i,j-1}^{BL}\right)' \\ \quad -\sum_{k=1}^{i} \left\{ \grave{b}_k \left(\grave{u}_{i-k,j-1}^{BL}\right)' - \grave{c}_k \grave{u}_{i-k,j}^{BL} \right\}, & i \geq 1, \quad j \geq 1, \end{cases} \tag{3.32}$$

where the notation $\breve{b}_k(\breve{x}) = \breve{x}^k b^{(k)}(0)/k!$ etc., is used once more. The above equations are supplemented with the following boundary conditions (in order to satisfy (2.2)):

$$\begin{aligned} &\breve{u}_{i,j}^{BL}(0) = -u_{i,j}(0), \\ &\grave{u}_{i,j}^{BL}(0) = -u_{i,j}(1), \\ &\lim_{\breve{x} \to \infty} \breve{u}_{i,j}^{BL}(\breve{x}) = 0, \quad \lim_{\grave{x} \to \infty} \grave{u}_{i,j}^{BL}(\grave{x}) = 0. \end{aligned} \tag{3.33}$$

The following result is established in a completely analogous way as in the previous cases (cf. Section 3.1).

Lemma 3.7. *Let $u_{i,j}$ be defined by* (3.30) *and assume* (2.3) *holds. Then there exist positive constants C, K_1, K_2, independent of $\varepsilon_1, \varepsilon_2$ such that*

$$\|u_{i,j}^{(n)}\|_{\infty,I} \leq Cn! K_1^n i! K_2^i, \quad \forall n \in \mathbb{N}.$$

Next we consider the boundary layers. The following result was shown in [6].

Proposition 3.1 ([6, Lemma 7.3.6]). *Let $\lambda \in \mathbb{C}$ with $Re(\lambda)>0, Re(\lambda)^2>0$. Let F be an entire function satisfying, for some $C_F>0$, $j\in\mathbb{N}_0$, $q\geq (j+1/2)/|\lambda|>0$,*

$$|F(z)|\leq C_F e^{-Re(\lambda z)}(q+|z|)^j, \quad \forall z\in\mathbb{C}.$$

Let $\alpha\in\mathbb{C}$ and let $v:(0,\infty)\to\mathbb{C}$, be the solution of the problem

$$-v''+\lambda^2 v=F \quad on \ \ (0,\infty), \quad v(0)=g, \quad \lim_{x\to\infty} v(x)=0.$$

Then v can be extended to an entire function (denoted again by v), which satisfies

$$|v(z)|\leq \left[C_F\frac{1}{|\lambda|}(q+|z|)^{j+1}(j+1)^{-1}+|g|\right]e^{-Re(\lambda z)}, \quad \forall z\in\mathbb{C}.$$

Using the above we may prove the following lemma.

Lemma 3.8. *Let $\check{u}_{i,j}^{BL}, \grave{u}_{i,j}^{BL}$ be defined by* (3.31), (3.32)*, respectively. Then there exist positive constants $C,\check{\gamma},\check{K},\grave{\gamma},\grave{K},\beta$, depending only on the data such that $\forall n\in\mathbb{N}$,*

$$\begin{aligned}\left|\left(\check{u}_{i,j}^{BL}\right)^{(n)}(x)\right| &\leq C\check{\gamma}^{i+j}(i+j)!\check{K}^n\varepsilon_1^{-n/2}e^{-\beta x/\sqrt{\varepsilon_1}}, \qquad \forall i,j\geq 0,\\ \left|\left(\grave{u}_{i,j}^{BL}\right)^{(n)}(x)\right| &\leq C\grave{\gamma}^{i+j}(i+j)!\grave{K}^n\varepsilon_1^{-n/2}e^{-\beta(1-x)/\sqrt{\varepsilon_1}}, \quad \forall i,j\geq 0,\end{aligned}$$

where $\beta=\min\{\check{c}_0,\grave{c}_0\}$.

Proof. First we show an estimate similar to (3.7), by using induction on i,j and Proposition 3.1. If $n>0$, the proof is almost identical to that of Lemma 3.6, utilizing Cauchy's integral theorem for derivatives. The details appear in [14]. □

We then define, for some $M\in\mathbb{N}$,

$$\begin{aligned} u_M(x) &= \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_1^{i/2}\left(\frac{\varepsilon_2}{\sqrt{\varepsilon_1}}\right)^j u_{i,j}(x),\\ \check{u}_M^{BL}(\check{x}) &= \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_1^{i/2}\left(\frac{\varepsilon_2}{\sqrt{\varepsilon_1}}\right)^j \check{u}_{i,j}^{BL}(\check{x}),\\ \grave{u}_M^{BL}(\grave{x}) &= \sum_{i=0}^{M}\sum_{j=0}^{M}\varepsilon_1^{i/2}\left(\frac{\varepsilon_2}{\sqrt{\varepsilon_1}}\right)^j \grave{u}_{i,j}^{BL}(\grave{x}),\end{aligned}$$

and we have the following decomposition:

$$u = u_M + \check{u}_M^{BL} + \grave{u}_M^{BL} + r_M. \tag{3.34}$$

The theorem that follows is the analog of Theorem 3.1 and its proof is almost identical. Nevertheless, it is worth commenting on the fact that ε_2 does not appear in the statement of Theorem 3.3. In the regime $\varepsilon_2 \ll \varepsilon_1$, the perturbation in the first order term is a regular perturbation, and as such benigne. As a result, its lack of presence in Theorem 3.3 is not an issue.

Theorem 3.3. *Assume* (2.3), (2.4) *hold. Then there exist positive constants* $K_1, \check{K}, \grave{K}, K_2$ *and* δ, *independent of* $\varepsilon_1, \varepsilon_2$, *such that the solution* u *of* (2.1)-(2.2) *can be decomposed as in* (3.34), *with*

$$\begin{aligned}
&\left\| u_M^{(n)} \right\|_{\infty,I} \lesssim n! K_1^n, && \forall n \in \mathbb{N}_0, \\
&\left| \left(\check{u}_M^{BL} \right)^{(n)} (x) \right| \lesssim \check{K}^n \varepsilon_1^{-n/2} e^{-\beta x/\sqrt{\varepsilon_1}}, && \forall n \in \mathbb{N}_0, \\
&\left| \left(\grave{u}_M^{BL} \right)^{(n)} (x) \right| \lesssim \grave{K}^n \varepsilon_1^{-n/2} e^{-\beta(1-x)/\sqrt{\varepsilon_1}}, && \forall n \in \mathbb{N}_0, \\
&\| r_M \|_{\infty,\partial I} + \| r_M \|_{0,I} + \varepsilon_1^{1/2} \| r_M' \|_{0,I} \lesssim e^{-\delta/\sqrt{\varepsilon_1}},
\end{aligned}$$

where M *is chosen so that* $\sqrt{\varepsilon_1} K_2 M < 1$, *and* $\beta = \min\{\check{c}_0, \grave{c}_0\}$.

3.4 On the transition between regimes

As a final question, we would like to see what happens when we fix $\varepsilon_1 \ll 1$ and consider $\varepsilon_2 \in (0,1]$. In Fig. 1 we show the solution of the BVP

$$\begin{aligned}
&-0.005 u''(x) + \varepsilon_2 u'(x) + u(x) = 1, \quad x \in (0,1), \\
&u(0) = u(1) = 0
\end{aligned}$$

for $\varepsilon_2 \in (0,1]$. The figure shows that the transition between regimes appears seamless, in the following sense: as ε_2 takes on values in $(0,1]$, the solution u smoothly moves from one regime to the other, based on the relationship between ε_2 and (the fixed, but small) ε_1. In particular

- If $\varepsilon_2 = 1$, then we have a convection-diffusion problem, and we have a layer of width $\mathcal{O}(\varepsilon_1)$ at the outflow boundary. (This is clearly visible in Fig. 1, on the right.)

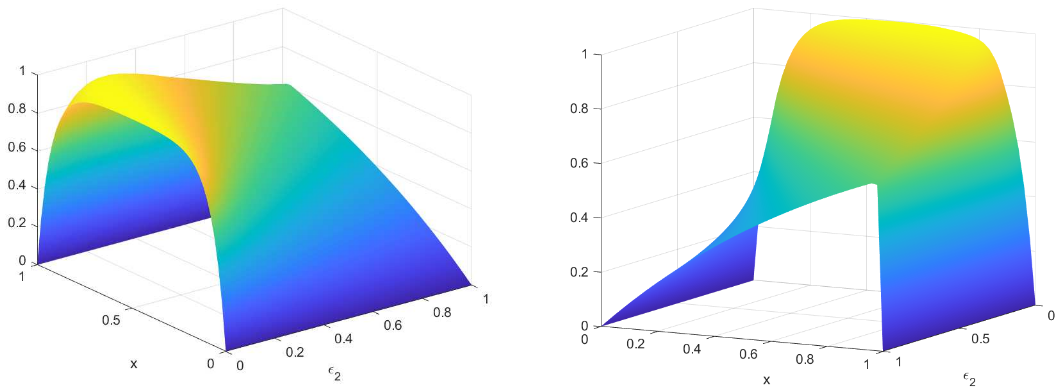

Figure 1: The solution $u(x)$ as a function of x and ε_2 (different viewing angles).

- If $1>\varepsilon_2>\sqrt{\varepsilon_1}$, then we have a reaction-convection-diffusion problem, with layers of different width at each endpoint.
- If $0<\varepsilon_2\leq\sqrt{\varepsilon_1}$, then we have a reaction-diffusion problem, with layers of width $\mathcal{O}(\sqrt{\varepsilon_1})$ at each endpoint. (See Fig. 1, on the left.)

The error bounds of the previous sections allow us to state the following.

Proposition 3.2. *Let $\varepsilon_1\ll 1$ be fixed, and let u be the solution of* (2.1)-(2.2) *under the assumption* (2.3)*. Then for any $\varepsilon_2\in(0,1)$, there exist positive constants K_1,K_2,K_3,δ, independent of $\varepsilon_1,\varepsilon_2$ such that*

$$u=u_S+u_{BL}^{\pm}+u_R.$$

The smooth part u_S, satisfies for any $n\in\mathbb{N}$, $\varepsilon_2\in(0,1)$,

$$\|u_S^{(n)}\|_{\infty,I}\lesssim K_1^n n!.$$

The boundary layer parts $u_{BL}^{\pm}$, satisfy for any $n\in\mathbb{N}$ and

- $\forall\varepsilon_2\in(0,\sqrt{\varepsilon_1}],x\in\overline{I}$,
$$\left|(u_{BL}^{\pm})^{(n)}(x)\right|\lesssim K_2^n n!\varepsilon_1^{-n/2}e^{-\beta\, dist\,(x,\partial I)/\sqrt{\varepsilon_1}},$$
- $\forall\varepsilon_2\in(\sqrt{\varepsilon_1},1),x\in\overline{I}$,
$$\left|(u_{BL}^{-})^{(n)}(x)\right|\lesssim K_2^n n!\varepsilon_2^{-n}e^{-\beta x/\varepsilon_2},$$
$$\left|(u_{BL}^{+})^{(n)}\right|\lesssim K_2^n n!\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{-n}e^{-\beta(1-x)\varepsilon_2/\varepsilon_1}.$$

The remainder u_R, satisfies for any $n \in \mathbb{N}$, $\varepsilon_2 \in (0,1)$,

$$\|u_R\|_{\infty,\partial I} + \|u_R\|_{0,I} + \min\{\varepsilon_2, \sqrt{\varepsilon_1}\}\|u_R'\|_{0,I} \lesssim \max\left\{e^{-\delta\varepsilon_2/\varepsilon_1}, e^{-\delta/\varepsilon_2}\right\}.$$

4 Conclusions

We considered a two-point, singularly perturbed, reaction-convection-diffusion problem with analytic input data, and we derived regularity results for its solution. Based on the relationship between the singular perturbation parameters, the problem becomes convection-diffusion, reaction-diffusion or reaction-convection-diffusion, as shown in Table 1. We provided estimates for all three cases (regimes), which reveal the analytic nature of the solution and give derivative bounds which are explicit in the differentiation order as well as the singular perturbation parameters. Such estimates are necessary for the construction and analysis of high order numerical methods, such as *hp* FEM (see, e.g. [15]).

References

[1] M. Brdar and H. Zarin, *A singularly perturbed problem with two parameters on a Bakhvalov-type mesh*, J. Comp. Appl. Math. 292 (2016), 307–319.

[2] J. L. Garcia, E. O'Riordan, and M. L. Pickett, *A parameter robust high order numerical method for a singularly perturbed two parameter problem*, Appl. Num. Math. 56 (2006), 962–980.

[3] T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics, Vol. 1985, Springer-Verlag, 2010.

[4] T. Linß and H. G. Roos, *Analysis of a finite difference scheme for a singularly perturbed problem with two small parameters*, Comput. Meth. Appl. Math. 289 (2004), 355–366.

[5] J. M. Melenk, *On the robust exponential convergence of hp finite element methods for problems with boundary layers*, IMA J. Num. Anal. 17 (1997), 577–601.

[6] J. M. Melenk, *hp-Finite Element Methods for Singular Perturbations*, Lecture Notes in Mathematics, Vol. 1796, Springer-Verlag, 2002.

[7] J. M. Melenk and C. Schwab, *An hp finite element method for convection-diffusion problems in one-dimension*, IMA J. Num. Anal. 19 (1999), 425–453.

[8] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Fitted Numerical Methods Singular Perturbation Problems*, World Scientific, 1996.

[9] K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Applied Mathematics and Mathematical Computation, Vol. 12, Chapman & Hall, 1996.

[10] R. E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag, 1991.

[11] E. O'Riordan, M. L. Pickett, and G. I. Shishkin, *Singularly perturbed problems modelling reaction-convection-diffusion processes*, Comput. Meth. Appl. Math. 3 (2003), 424–442.

[12] H.-G. Roos, M. Stynes, and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations. Convection-Diffusion-Reaction and Flow Problems*, Springer Series in Computational Mathematics, Vol. 24, Springer-Verlag, 2008.

[13] H.-G. Roos and Z. Uzelac, *The SDFEM for a convection-diffusion problem with two small parameters*, Comput. Methods Appl. Math. 3 (2003), 443–458.

[14] I. Sykopetritou, *An hp Finite Element Method for a Second Order Singularly Perturbed Boundary Value Problem with Two Small Parameters*, Ph.D. Thesis, Department of Mathematics & Statistics, University of Cyprus, expected 2024.

[15] I. Sykopetritou and C. Xenophontos, *An hp finite element method for a convection-reaction-diffusion singularly perturbed problem with two small parameters*, Int. J. Numer. Anal. Model. 18 (2021), 481–499.

[16] Lj. Teofanov and H.-G. Roos, *An elliptic singularly perturbed problem with two parameters I: Solution decomposition*, J. Comput. Appl. Math. 206 (2007), 1802–1097.

[17] Lj. Teofanov and H.-G. Roos, *An elliptic singularly perturbed problem with two parameters II: Robust finite element solution*, J. Comput. Appl. Math. 212 (2008), 374–389.