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A Mass-Preserving Characteristic Finite Difference Method For Miscible Displacement Problem

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Abstract. In this article, a new characteristic finite difference method is developed for solving miscible displacement problem in porous media. The new method combines the characteristic technique with mass-preserving interpolation, not only keeps mass balance but also is of second-order accuracy both in time and space. Numerical results are presented to confirm the convergence and the accuracy in time and space.

AMS subject classifications: 65M06, 65M25

Key words: The method of characteristics, mass-preserving, finite difference, miscible displacement problem.

1 Introduction

During recent years, the research of oil reservoir numerical simulation has been an important field in modern computational mathematics. In this field, two-phase flow displacement (water and oil) is one of the most important basic problems. In this article, we will consider to construct a new numerical method for the following incompressible miscible displacement problem in porous media, which is governed by a nonlinear coupled system of partial differential equations: the pressure is governed by an elliptic equation and the concentration is governed by a convection-diffusion equation (see [1–3]):

$$\begin{cases} \nabla\cdot\mathbf{u}=g(\mathbf{x},t), \quad \mathbf{u}=-\dfrac{\kappa(\mathbf{x})}{\mu(c)}\nabla p=-r(\mathbf{x},c)\nabla p, & (\mathbf{x},t)\in\Omega\times(0,T], \\ \phi(\mathbf{x})\dfrac{\partial c}{\partial t}+\nabla\cdot(\mathbf{u}c-D\nabla c)=\tilde{c}g(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times(0,T], \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in R^d, $(d=1,2,3)$, $p(\mathbf{x},t)$ denotes the pressure, $\mathbf{u}$ is the Darcy velocity, $c(\mathbf{x},t)$ is the relative concentration, $\kappa(\mathbf{x})$ is the permeability of strata, $\mu(c)$ is the viscosity of the fluid mixture, $\phi(\mathbf{x})$ is the porosity of the rock, $g(\mathbf{x},t)$ is the external flow rate, which is positive if fluid is being injected, the concentration $\tilde{c}$ in the source term is the injected concentration c_ω if $g(\mathbf{x},t)\geq 0$ and is the resident concentration c if $g(\mathbf{x},t)<0$. Furthermore, a compatibility condition $\int_\Omega p d\mathbf{x}=0$, $0\leq t\leq T$ must be imposed to determine the pressure.

The pressure equation is elliptic and easily handled, but the concentration equation is parabolic and normally convection-dominated. It is well known that the standard finite difference method and Galerkin finite element method applied to the convection-dominated problems do not work well, and produce excessive numerical diffusion or nonphysical oscillation. A variety of numerical techniques were introduced to obtain better approximations, such as, Yuan etc. proposed the modified method of characteristic finite element method (MMOC) in [4–9], and Russell proposed the Eulerian-Lagrangian localized adjoint method (ELLAM) in [10]. Moreover, Yang proposed least-squares mixed finite element method in [11], and Yuan, Liang and Rui etc. proposed the characteristic finite difference schemes, the modified method of upwind with finite difference fractional steps procedures, see [12–14]. Each of the above methods has its advantages and disadvantages. Upstream-weighted method tends to introduce an excessive amount of numerical diffusion near the sharp fronts into the solution. Streamline diffusion method and least-squares mixed finite element method reduce the amount of diffusion but add a user-defined amount biased in the direction of the streamline. ELLAM conserves mass locally but it is difficult to evaluate the resulting integrals. Explicit characteristic and Godunov schemes require a CFL time-step constraint. The MMOC-Galerkin scheme has much smaller numerical diffusion than those of standard Galerkin methods, and can be used with a larger time step, with corresponding improvement in efficiency and without cost in accuracy. But it fails to keep mass balance.

In [15, 16], Liang and Fu proposed a new efficient high-order mass-conservative finite difference method for the advection-dominated transport problem. This algorithm combines the characteristic technique with the conservative interpolation technique as in [17, 18]. It does not only keep mass balance but also does well in the advection-dominated diffusion problem. And then, Fu combined block-centered finite difference method with this technique for convection domminated diffusion equations in [19]. In this article, our main purpose is to use the similar technique as in [15, 16] to construct a new combined numerical scheme for incompressible miscible displacement problem. In this new algorithm, the time second-order splitting technique is used to obtain a second-order mass-preserving characteristic finite difference (MPC-FD) method for the concentration. Based on the characteristic form of the advection-diffusion equations tracking back along the characteristic curve, the integrals over the tracking cells at the previous time level are treated by the conservative interpolation distribution and the diffusion terms are approximated by averaging along the characteristics. Meanwhile, the space second-order finite difference scheme is used for the pressure and Darcy velocity.

The outline of this article is organized as follows. In Section 2, we first give the finite difference scheme for the velocity and pressure, and formulate the MPC-FD method for the concentration, and then we combine the finite difference method for miscible displacement problem with the method of characteristics to construct the new finite difference method. In Section 3, we give some numerical experiments to confirm the convergence of this method and the accuracy in time and space. Finally, we give a conclusion in Section 4.

2 The MPC-FD method for one dimensional problem

For convenience of analysis, we consider one dimensional model problem. So we rewrite the equation (1.1) as follows:

$$\begin{cases} u_x = g(x,t), & x\in\Omega, \quad t\in(0,T], \\ u = -k(x,c)p_x, & x\in\Omega, \quad t\in(0,T], \\ \phi(x)\dfrac{\partial c}{\partial t} + \dfrac{\partial(uc)}{\partial x} - \dfrac{\partial}{\partial x}\left(D(u)\dfrac{\partial c}{\partial x}\right) = q(x,t), & x\in\Omega, \quad t\in(0,T], \end{cases} \tag{2.1}$$

where $\Omega=[a,b]$, $k(x,c)=\kappa(x)/\mu(c)$, $q(x,t)$ is the given function. And the initial-boundary conditions are:

$$\begin{aligned} &c(x,t)|_{\partial\Omega} = u(x,t)|_{\partial\Omega} = 0, \qquad && c_x(x,t)|_{\partial\Omega} = 0, \\ &p(x,0) = p^0(x), && c(x,0) = c^0(x). \end{aligned}$$

Divide Ω by

$$a = x_{1/2} < x_{3/2} < \dots < x_{I-1/2} < x_{I+1/2} = b$$

and cells centers and sizes are defined by

$$\begin{aligned} &\Omega_i = [x_{i-1/2}, x_{i+1/2}], \quad x_i = \frac{1}{2}(x_{i-1/2} + x_{i+1/2}), \\ &\Delta x_i = x_{i+1/2} - x_{i-1/2}, \quad i = 1,2,\dots,I. \end{aligned}$$

Denote $\Omega_h = \{\Omega_i,\ i=1,2,\dots,I\}$, here we consider an uniform mesh Ω_h with $\Delta x_i = h$. Let N_t be a positive integer, $\triangle t = T/N_t$ is a time step size, $t^n = n\triangle t$.

2.1 The approximate scheme for the pressure and Darcy velocity

In this part, we consider the finite difference approximations of the pressure and the Darcy velocity. Denote

$$K_i^n = \frac{1}{2}\left[k(x_{i-1/2},C_{i-1/2}^n)+k(x_{i+1/2},C_{i+1/2}^n)\right], \tag{2.2a}$$

$$k_i^n = \frac{1}{2}\left[k(x_{i-1/2},c_{i-1/2}^n)+k(x_{i+1/2},c_{i+1/2}^n)\right], \tag{2.2b}$$

$$\partial_{\bar{x}}(K\partial_x P)_{i+1/2}^n = h^{-2}\left[K_{i+1}^n(P_{i+3/2}^n-P_{i+1/2}^n)-K_i^n(P_{i+1/2}^n-P_{i-1/2}^n)\right], \tag{2.2c}$$

$$\partial_{\bar{x}}(k\partial_x P)_{i+1/2}^n = h^{-2}\left[k_{i+1}^n(P_{i+3/2}^n-P_{i+1/2}^n)-k_i^n(P_{i+1/2}^n-P_{i-1/2}^n)\right], \tag{2.2d}$$

where $\partial_{\bar{x}}$, ∂_x represent the space forward and backward difference quotient, respectively, c and p are the exact solutions, C and P are approximate solutions. Then, based on (2.2) we give the following finite difference approximation of the first and second equations of (2.1).

Scheme I. For a given function c, seek (P^n,U^n) $(n=0,1,2,\cdots,N_t)$, such that

$$\partial_{\bar{x}}(k\partial_x P)_{i+1/2}^n = g(x_{i+1/2},t^n), \qquad 0\le i\le I, \tag{2.3a}$$

$$U_{i+1/2}^n - U_{i-1/2}^n = \int_{\Omega_i} g(x,t^n)dx, \qquad 0\le 0\le I-1. \tag{2.3b}$$

Assume that $k(x,c)$ is nonsingular function, such that

$$k(x,c)\ge k_*>0,$$

where k_* is a known positive constant. Then problem (2.3) exists a unique solution.

2.2 The approximate scheme for the concentration

In this part, we consider the time second-order mass-preserving characteristic finite difference method for the concentration. Firstly we denote the characteristic direction by τ in the each time cell $[t^n,t^{n+1}]$. The characteristic cure can be denoted by $X(\tau;x,t^{n+1})$. For convenience of analysis, denote $X_x^{n+1}(\tau)=X(\tau;x,t^{n+1})$ and $X_{i-1/2}^{n+1}(\tau)=X(\tau;x_{i-1/2},t^{n+1})$. Then the cure $X_x^{n+1}(\tau)$ from the point (x,t^{n+1}) satisfies

$$\begin{cases} \dfrac{dX_x^{n+1}(\tau)}{d\tau} = u(X_x^{n+1}(\tau),\tau)/\phi(x), & \tau\in[t^n,t^{n+1}], \\ X_x^{n+1}(t^{n+1})=x, & \text{when } \ \tau=t^{n+1}. \end{cases} \tag{2.4}$$

Let $\bar{x}^n = X^{n+1}(t^n)$ be the intersection point of the characteristic with level $t=t^n$, and let

$$\Omega_i(t)=[X_{i-1/2}^{n+1}(t),X_{i+1/2}^{n+1}(t)], \quad \overline{\Omega}_i(t^n)=[X_{i-1/2}^{n+1}(t^n),X_{i+1/2}^{n+1}(t^n)]=[\bar{x}_{i-1/2}^n,\bar{x}_{i+1/2}^n].$$

In order to obtain the second-order accuracy in time, we use a second-order Runge-Kutta method to solve numerically $\bar{x}^n_{i+1/2}$ at $t=t^n$ from (2.4)

$$\overline{u}^{n+1/2}(x_{i+1/2})=\overline{u}^{n+1/2}_{i+1/2}=\frac{1}{2}[u^{n+1}_{i+1/2}+u(x_{i+1/2}-\Delta t u^{n+1}_{i+1/2},t^n)], \tag{2.5a}$$

$$\bar{x}_{i+1/2}\approx x_{i+1/2}-\overline{u}^{n+1/2}_{i+1/2}\Delta t/\phi_{i+1/2}, \tag{2.5b}$$

where $u(x_{i+1/2},t^{n+1})$ is denoted by $u^{n+1}_{i+1/2}$.

Applying the Leibniz rule, we have

$$\frac{d}{dt}\int_{\Omega_i(t)}\Big(\phi(x)c\Big)dx=\phi(x)\frac{d}{dt}\int_{\Omega_i(t)}cdx=\int_{\Omega_i(t)}\Big[\phi(x)\frac{\partial c}{\partial t}+\frac{\partial}{\partial x}\Big(uc\Big)\Big]dx.$$

Then, integrating the third equation of (2.1) leads to

$$\int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)}\Big[\phi(x)\frac{\partial c}{\partial t}+\frac{\partial}{\partial x}\Big(uc\Big)-\frac{\partial}{\partial x}\Big(D(u)\frac{\partial c}{\partial x}\Big)\Big]dxdt=\int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)}q(x,t)dxdt. \tag{2.6}$$

Hence, using the characteristic equation (2.4), we can get

$$\begin{aligned}&\int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)}\Big[\phi(x)\frac{\partial c}{\partial t}+\frac{\partial}{\partial x}\Big(uc\Big)\Big]dxdt\\&=\int_{\Omega_i}\phi(x)c(x,t^{n+1})dx-\int_{\bar{\Omega}_i(t^n)}\phi(x)c(x,t^n)dx.\end{aligned} \tag{2.7}$$

Next, we deal with the third term on the left-hand-side of (2.6). We have

$$\begin{aligned}&\int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)}\frac{\partial}{\partial x}\Big(D(u)\frac{\partial c}{\partial x}\Big)dxdt\\&=\int_{t^n}^{t^{n+1}}\Big[\Big(D(u)\frac{\partial c}{\partial x}\Big)(X^{n+1}_{i+1/2}(t),t)-\Big(D(u)\frac{\partial c}{\partial x}\Big)(X^{n+1}_{i-1/2}(t),t)\Big]dt.\end{aligned} \tag{2.8}$$

By applying the Leibniz rule again, and the characteristic equation (2.4), we get the local mass conservation finite difference formula of the concentration equation

$$\begin{aligned}&\int_{\Omega_i}\phi(x)c(x,t^{n+1})dx-\int_{\bar{\Omega}_i(t^n)}\phi(x)c(x,t^n)dx\\&\quad-\int_{t^n}^{t^{n+1}}\Big[\Big(D(u)\frac{\partial c}{\partial x}\Big)(X^{n+1}_{i+1/2}(t),t)-\Big(D(u)\frac{\partial c}{\partial x}\Big)(X^{n+1}_{i-1/2}(t),t)\Big]dt\\&=\int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)}q(x,t)dxdt.\end{aligned} \tag{2.9}$$

Denote $\phi(x_j)C_j$ the cell-averaged concentration at time t on the cell $\Omega_j=[x_{j-1/2},x_{i+1/2}]$

$$\phi(x_j)C_j\approx\frac{1}{h}\int_{\Omega_j}\phi(x)c(x,t)dx. \tag{2.10}$$

Based on (2.10), using the unknown C_j^{n+1} at time $t=t^{n+1}$, we approximate the first term on the left-hand-side of (2.9) as

$$\int_{\Omega_i} \phi(x)c(x,t^{n+1})dx \approx h(\phi C^{n+1})_i. \tag{2.11}$$

In order to conserve mass balance, we define a particular parabolic interpolation distribution $[R(\phi C^n)](x)$ to approximate $\phi(x)c(x,t^n)$ in $\overline{\Omega}_i^n$ by Piecewise Parabolic Method as in [17],

$$\begin{aligned}[R(\phi C^n)]_j(x) = &\widehat{(\phi C^n)}_{j-1/2} + \frac{x-x_{j-1/2}}{h}\Bigg\{\widehat{(\phi C^n)}_{j+1/2} - \widehat{(\phi C^n)}_{j-1/2} \\ &+6\left[\phi C_j^n - \frac{1}{2}(\widehat{(\phi C^n)}_{j-1/2} + \widehat{(\phi C^n)}_{j+1/2})\right]\frac{x_{j+1/2}-x}{h}\Bigg\},\end{aligned} \tag{2.12}$$

where $\widehat{(\phi C^n)}_{j-1/2}$ and $\widehat{(\phi C^n)}_{j+1/2}$ denote the estimated values at $x_{j-1/2}$ and $x_{j+1/2}$, which will be defined later. Apparently, $[R(\phi C^n)]_j(x)$ satisfies

$$\begin{cases} [R(\phi C^n)]_j(x_{j-1/2}) = \widehat{(\phi C^n)}_{j-1/2}, \\ [R(\phi C^n)]_j(x_{j+1/2}) = \widehat{(\phi C^n)}_{j+1/2}, \\ \displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} [R(\phi C^n)]_j(x)dx = h(\phi C^n)_j. \end{cases} \tag{2.13}$$

Using this definition, we can get the approximation of the second term on the left-hand-side of (2.9)

$$\begin{aligned}&\int_{\overline{\Omega}_i(t^n)} \phi(x)c(x,t^n)dx \approx \int_{\bar{x}_{i-1/2}}^{\bar{x}_{i+1/2}} [R(\phi C^n)](x)dx \\ =&\begin{cases} \displaystyle\int_{\bar{x}_{l-1/2}}^{x_{l+1/2}} [R(\phi C^n)]_l(x)dx + \sum_{j=l+1}^{m-1} h(\phi C^n)_j + \int_{x_{m-1/2}}^{\bar{x}_{i+1/2}} [R(\phi C^n)]_m(x)dx, & m \geq l+1, \\ \displaystyle\int_{\bar{x}_{i-1/2}}^{\bar{x}_{i+1/2}} [R(\phi C^n)]_l(x)dx, & m = l, \end{cases} \\ \triangleq& R_{h,\phi,\overline{\Omega}_i(C^n)},\end{aligned} \tag{2.14}$$

which is a summation upon many cells $\overline{\Omega}_i^n$ covers. l and $m \geq l+1$ are the cell indices associated with the segments in which $\bar{x}_{i+\frac{1}{2}}$ and $\bar{x}_{i-1/2}$ lie, in other words, $\bar{x}_{i+1/2} \in \Omega_m$ and $\bar{x}_{i-1/2} \in \Omega_l$.

Now, we give the definitions of $\widehat{(\phi C^n)}_{j-1/2}$ and $\widehat{(\phi C^n)}_{j+1/2}$. Using a cumulative mass function defined by

$$L(x) = \int_a^x \phi C(y,t)dy,$$

and taking the derivative with respect to x for $L(x)$, we have

$$\phi(x)C(x,t)=\frac{dL(x)}{dx}.$$

For the fixed point $x=x_{j+1/2}$, we have

$$\phi(x_{j+1/2})C(x_{j+1/2},t)=\frac{dL(x)}{dx}\Big|_{x_{j+1/2}}. \tag{2.15}$$

Next, in order to obtain the space second-order approximate scheme, we use the quadratic polynomial $P_2(x)$ to interpolate $L^n(x)$ with three points, and the three points as follows

$$(L^n_{j-1/2},x_{j-1/2}); \quad (L^n_{j+1/2},x_{j+1/2}); \quad (L^n_{j+3/2},x_{j+3/2}).$$

When $x=x_{j+k+1/2}$, $L^n(x)$ satisfies

$$L^n(x_{j+k+1/2})=\sum_{p\le j+k}(\phi C)_p h, \quad k=-1,0,1. \tag{2.16}$$

Thus, for $L(x)$, taking the derivative with respect to x , we have

$$\begin{aligned}\frac{dL(x)}{dx}=&\sum_{p\le j-1}(\phi C)_p\left(\frac{2x-x_{j+\frac{1}{2}}-x_{j+\frac{3}{2}}}{2h}\right)-\sum_{p\le j}(\phi C)_p\left(\frac{2x-x_{j-\frac{1}{2}}-x_{j+\frac{3}{2}}}{h}\right)\\&+\sum_{p\le j+1}(\phi C)_p\left(\frac{2x-x_{j-\frac{1}{2}}-x_{j+\frac{1}{2}}}{2h}\right).\end{aligned} \tag{2.17}$$

When $t=t^n$ and $x=x_{j+1/2}$, we have

$$\widehat{\phi C}^n_{j+1/2}=\frac{1}{2}\left[(\phi C^n)_j+(\phi C^n)_{j+1}\right], \quad j=1,2,\dots,I. \tag{2.18}$$

And (2.8) becomes as follows

$$\begin{aligned}&\int_{t^n}^{t^{n+1}}\left(D(u)\frac{\partial c}{\partial x}\right)(X^{n+1}_{i+1/2}(t),t)dt\\ \approx&\frac{\Delta t}{2}\left[D\left(u^{n+1}(x_{i+1/2})\right)\frac{\partial c^{n+1}}{\partial x}\Big|_{x_{i+1/2}}+D(u^n(\bar{x}_{i+1/2}))\frac{\partial c^n}{\partial x}\Big|_{\bar{x}_{i+1/2}}\right]\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\int_{t^n}^{t^{n+1}}\left(D(u)\frac{\partial c}{\partial x}\right)(X^{n+1}_{i-1/2}(t),t)dt\\ \approx&\frac{\Delta t}{2}\left[D\left(u^{n+1}(x_{i-1/2})\right)\frac{\partial c^{n+1}}{\partial x}\Big|_{x_{i-1/2}}+D(u^n(\bar{x}_{i-1/2}))\frac{\partial c^n}{\partial x}\Big|_{\bar{x}_{i-1/2}}\right].\end{aligned} \tag{2.20}$$

Using (2.17) and making derivation with respect to x, we have

$$\frac{\partial c}{\partial x}\Big|_x = \frac{d^2L}{dx^2}\Big|_x = \frac{C_{i+1}-C_i}{h}. \tag{2.21}$$

When $t=t^{n+1}$, using the quadratic polynomial $P_2(x)$ to approximate the flux $\frac{\partial c^{n+1}}{\partial x}\big|_{x_{i+1/2}}$ that interpolates $L^{n+1}(x)$ through three points $(L^{n+1}_{j+k+1/2}, x_{j+k+1/2})$, $k=-1,0,1$, thus we get the second order difference operator

$$\frac{\partial c^{n+1}}{\partial x}\Big|_{x_{i+1/2}} \approx \frac{C^{n+1}_{i+1}-C^{n+1}_i}{h} \triangleq \delta_x C^{n+1}_{i+1/2}. \tag{2.22}$$

We know that $\bar{x}_{i+1/2}$ and $\bar{x}_{i-1/2}$ are not at the regular mesh points, thus the flux $\frac{\partial c^n}{\partial x}\big|_{\bar{x}_{i+1/2}}$ is not easy to be treated. Then through four points

$$(L^n_{j-3/2}, x_{j-3/2}), \quad (L^n_{j-1/2}, x_{j-1/2}), \quad (L^n_{j+1/2}, x_{j+1/2}), \quad (L^n_{j+3/2}, x_{j+3/2}),$$

we can use cubic polynomial $P_3(x)$ to approximate the $L^n(x)$ as above interpolation method, thus we have

$$\frac{\partial c^n}{\partial x}\Big|_{\bar{x}_{i+1/2}} \approx [(\bar{\theta}_{i+1/2}-1)C^n_{m-1}+(-2\bar{\theta}_{i+1/2}+1)C^n_m+\bar{\theta}_{i+1/2}C^n_{m+1}]/h \triangleq \delta'_x C^n\Big|_{\bar{x}_{i+1/2}} \tag{2.23}$$

and

$$\frac{\partial c^n}{\partial x}\Big|_{\bar{x}_{i-1/2}} \approx [(\bar{\theta}_{i-1/2}-1)C^n_{m-1}+(-2\bar{\theta}_{i-1/2}+1)C^n_m+\bar{\theta}_{i-1/2}C^n_{m+1}]/h \triangleq \delta'_x C^n\Big|_{\bar{x}_{i-1/2}}, \tag{2.24}$$

where $\bar{\theta}_{i+1/2}=(\bar{x}_{i+1/2}-x_{m+1/2})/h$, $\bar{\theta}_{i-1/2}=(\bar{x}_{i-1/2}-x_{m+1/2})/h$.

From (2.9)-(2.24), we come up with the mass conservative characteristic finite difference scheme for the concentration equation.

Scheme II. For a given u, seek C^{n+1} $(n=0,1,\ldots,N_t-1)$, such that

$$\begin{aligned}
&\Big(h\phi(x_i)C^{n+1}_i - R_{h,\phi,\bar{\Omega}_i(C^n)}\Big) - \frac{\Delta t}{2}\Big[\Big(D(u^{n+1}_{i+1/2})\delta_x C^{n+1}_{i+1/2} + D(u^n_{\bar{x}_{i+1/2}})\delta'_x C^n\Big|_{\bar{x}_{i+1/2}}\Big) \\
&\qquad - \Big(D(u^{n+1}_{i-1/2})\delta_x C^{n+1}_{i-1/2} + D(u^n_{\bar{x}_{i-1/2}})\delta'_x C^n\Big|_{\bar{x}_{i-1/2}}\Big)\Big] \\
&= \int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)} q(x,t)dxdt \quad i=1,2,\ldots,I,
\end{aligned} \tag{2.25}$$

with the initial condition

$$C^0_{i+1/2} = c^0(x_{i+1/2}), \quad 0\le i\le I. \tag{2.26}$$

Next, we can show the mass conservation property of Scheme II. Assume that the velocity $u:\Omega\times(0,T)\to R$ satisfies

$$u\in C^0(W^{1,\infty}(\Omega)), \tag{2.27}$$

then we have the following theorem.

Theorem 2.1. *Under hypothesis* (2.27)*, Scheme II keeps mass balance.*

Proof. Summing the equation (2.25) from $i=1$ to I leads to

$$\begin{aligned}&\sum_{i=1}^{I}\Big(h\phi(x_i)C_i^{n+1}-R_{h,\phi,\bar{\Omega}_i(C^n)}\Big)-\sum_{i=1}^{I}\frac{\Delta t}{2}\Big[\Big(D(u_{i+1/2}^{n+1})\delta_x C_{i+1/2}^{n+1}+D(u_{\bar{x}_{i+1/2}}^{n})\delta_x' C^n\Big|_{\bar{x}_{i+1/2}}\Big)\\&\quad-\Big(D(u_{i-1/2}^{n+1})\delta_x C_{i-1/2}^{n+1}+D(u_{\bar{x}_{i-1/2}}^{n})\delta_x' C^n\Big|_{\bar{x}_{i-1/2}}\Big)\Big]\\&=\int_{t^n}^{t^{n+1}}\int_{\Omega}q(x,t)dxdt.\end{aligned}\tag{2.28}$$

Under the hypothesis (2.27), boundary conditions, $\bar{x}_{1/2}=a$ and $\bar{x}_{I+1/2}=b$, (2.14) lead to

$$\sum_{i=1}^{I}h\phi(x_i)C_i^n=\sum_{i=1}^{I}R_{h,\phi,\bar{\Omega}_i(C^n)}.\tag{2.29}$$

Notice that it holds

$$\begin{aligned}&\sum_{i=1}^{I}\Big[\Big(D(u_{i+1/2}^{n+1})\delta_x C_{i+1/2}^{n+1}+D(u_{\bar{x}_{i+1/2}}^{n})\delta_x' C^n\Big|_{\bar{x}_{i+1/2}}\Big)\\&\qquad-\Big(D(u_{i-1/2}^{n+1})\delta_x C_{i-1/2}^{n+1}+D(u_{\bar{x}_{i-1/2}}^{n})\delta_x' C^n\Big|_{\bar{x}_{i-1/2}}\Big)\Big]\\&=\Big[\Big(D(u_{I+1/2}^{n+1})\delta_x C_{I+1/2}^{n+1}+D(u_{\bar{x}_{I+1/2}}^{n})\delta_x' C^n\Big|_{\bar{x}_{I+1/2}}\Big)\\&\qquad-\Big(D(u_{1/2}^{n+1})\delta_x C_{1/2}^{n+1}+D(u_{\bar{x}_{1/2}}^{n})\delta_x' C^n\Big|_{\bar{x}_{1/2}}\Big)\Big]=0.\end{aligned}\tag{2.30}$$

From (2.28)-(2.30), we have

$$\sum_{i=1}^{I}h\phi(x_i)C_i^{n+1}-\sum_{i=1}^{I}h\phi(x_i)C_i^n=\int_{t^n}^{t^{n+1}}\int_{\Omega}\tilde{c}g(x,t)dxdt.$$

Obviously, Scheme II (2.25) keeps mass balance. □

2.3 The combined method for miscible displacement problem

Now, we propose the mass-preserving characteristic finite difference method for incompressible miscible displacement problem. Replacing $u_{i+1/2}^{n+1}$ and $u_{i+1/2}^{n}$ by $U_{i+1/2}^{n+1}$ and $U_{i+1/2}^{n}$ in (2.5), we modify the definition of $\bar{x}_{i+1/2}$ as follows

$$\bar{x}_{i+1/2}=x_{i+1/2}-\overline{U}_{i+1/2}^{n+1/2}\Delta t/\phi_{i+1/2},\tag{2.31a}$$

$$\overline{U}_{i+1/2}^{n+1/2}=\frac{1}{2}[U_{i+1/2}^{n+1}+U_{i+1/2}^{n}-\Delta t g_{i+1/2}^{n}U_{i+1/2}^{n+1}].\tag{2.31b}$$

Based on Scheme I and Scheme II, we construct the new characteristic finite difference method as follows:

Scheme III. For the given initial approximate value C^0, such that

$$C^0_{i+1/2}=c^0(x_{i+1/2}),\quad 0\le i\le I,$$

seek (P^n,U^n), $(n=0,1,\cdots,N_t)$, such that

$$h^{-2}\left[K^n_{i+1}(P^n_{i+3/2}-P^n_{i+1/2})-K^n_i(P^n_{i+1/2}-P^n_{i-1/2})\right]=g(x_{i+1/2},t^n),\qquad 0\le i\le I,\tag{2.32a}$$

$$U^n_{i+1/2}-U^n_{i-1/2}=\int_{\Omega_i}g(x,t^n)dx,\qquad 0\le i\le I,\tag{2.32b}$$

and C^{n+1}, $(n=0,1,\cdots,N_t-1)$ such that

$$\begin{aligned}&\Big(h\phi(x_i)C^{n+1}_i-R_{h,\phi,\bar{\Omega}_i(C^n)}\Big)-\frac{\Delta t}{2}\Big[\Big(D(U^{n+1}_{i+1/2})\delta_x C^{n+1}_{i+1/2}+D(U^n_{\bar{x}_{i+1/2}})\delta'_x C^n\Big|_{\bar{x}_{i+1/2}}\Big)\\&\quad-\Big(D(U^{n+1}_{i-1/2})\delta_x C^{n+1}_{i-1/2}+D(U^n_{\bar{x}_{i-1/2}})\delta'_x C^n\Big|_{\bar{x}_{i-1/2}}\Big)\Big]\\&=\int_{t^n}^{t^{n+1}}\int_{\Omega_i(t)}q(x,t)dxdt,\quad 0\le i\le I,\end{aligned}\tag{2.33}$$

where $R_{h,\phi,\bar{\Omega}_i(C^n)}$ is defined by (2.14) and $U^n_{\bar{x}_{i-1/2}}$ can be approximated by the following formula

$$U^n_{\bar{x}_{i+1/2}}-U^n_{\bar{x}_{i-1/2}}=\int_{\Omega_{i(t^n)}}g(x,t^n)dx.$$

3 Numerical experiments

In this section, we will present some numerical results to observe the performance of the time second-order mass-preserving characteristic finite difference method.

3.1 One dimensional case

Define L^∞-norm and L^2-norm as follows:

$$E^n_\infty=\max_i\{|c(x_i,t^n)-C^n_i|\},\tag{3.1a}$$

$$E^n_2=\sqrt{\sum_i\Delta x(c(x_i,t^n)-C^n_i)^2}.\tag{3.1b}$$

We take c satisfying the following initial-boundary conditions

$$c(x,0)=\exp\Big(-\frac{(x-x_0)^2}{2\sigma^2}\Big),\quad x\in\Omega,\tag{3.2a}$$

$$c(0,t)=f(t)=0,\tag{3.2b}$$

$$\frac{\partial c}{\partial x}(1,t)=0.\tag{3.2c}$$

Experiment I. In this experiment, we will consider the case: u satisfies the non-homogeneous boundary condition. Set $\Omega=[0,1]$, $[0,T]=[0,0.3]$, $x_0=0.15$, $\phi(x)=1$, $k=1$ and $\sigma=0.03$. For the fixed space step $\Delta x=\frac{1}{2000}$ and $\Delta x=\frac{1}{4000}$, we give the L^∞ and L^2 error results with different time steps, Darcy velocity u and diffusion coefficients D, see Tables 1-4. And then, for the fixed time increment $\Delta t=\frac{1}{90}$ and $\Delta t=\frac{1}{100}$, we give the errors and ratios with different space steps and diffusion coefficients D, see Tables 5-6. These numerical results show that our method keeps second-order accuracy both in time and space.

Table 1: Errors and ratios in time with $u=1$, $D=10^{-2}$, $x_0=0.15$, $\sigma=0.03$ and $\Delta x=\frac{1}{2000}$.

	Δt	1/40	1/50	1/60	1/70	1/80
C-FD	E_∞	4.6645e-02	3.7667e-02	3.1564e-02	2.7144e-02	2.3797e-02
	Ratio	-	0.9580	0.9697	0.9785	0.9857
	E_2	1.7062e-02	1.3790e-02	1.1565e-02	9.9541e-03	8.7331e-03
	Ratio	-	0.9541	0.9650	0.9733	0.9800
MPC-FD	E_∞	2.5570e-04	1.6356e-04	1.1337e-04	8.3060e-05	6.3363e-05
	Ratio	-	2.0022	2.0102	2.0184	2.0276
	E_2	8.7538e-05	5.5809e-05	3.8723e-05	2.8544e-05	2.2200e-05
	Ratio	-	2.0046	2.0173	1.9784	1.8822

Table 2: Errors and ratios in time with $u=1$, $D=10^{-2}$, $x_0=0.15$, $\sigma=0.03$ and $\Delta x=\frac{1}{4000}$.

	Δt	1/40	1/50	1/60	1/70	1/80
C-FD	E_∞	4.6871e-02	3.7899e-02	3.1797e-02	2.7380e-02	2.4034e-02
	Ratio	-	0.9523	0.9627	0.9703	0.9761
	E_2	1.7143e-02	1.3873e-02	1.1650e-02	1.0039e-02	8.8819e-03
	Ratio	-	0.9484	0.9580	0.9650	0.9705
MPC-FD	E_∞	2.5679e-04	1.6465e-04	1.1446e-04	8.4144e-05	6.4444e-05
	Ratio	-	1.9917	1.9943	1.9961	1.9975
	E_2	8.7908e-05	5.6158e-05	3.9129e-05	2.9026e-05	2.2543e-05
	Ratio	-	2.0083	1.9816	1.9376	1.8928

Table 3: Errors and ratios in time with $u=1$, $D=0$, $x_0=0.15$, $\sigma=0.03$ and $\Delta x=\frac{1}{4000}$.

	Δt	1/40	1/50	1/60	1/70	1/80
C-FD	E_∞	4.8537e-02	3.8971e-02	3.2541e-02	2.7923e-02	2.4444e-02
	Ratio	-	0.9837	0.9889	0.9930	0.9963
	E_2	1.7921e-02	1.4390e-02	1.2015e-02	1.0310e-02	9.0256e-03
	Ratio	-	0.9837	0.9889	0.9930	0.9963
MPC-FD	E_∞	1.6693e-04	8.2151e-05	5.6848e-05	3.5310e-05	2.6922e-05
	Ratio	-	2.1775	2.0194	2.0893	2.0312
	E_2	3.4812e-05	1.8792e-05	1.2514e-05	9.3200e-06	7.5149e-06
	Ratio	-	2.5630	2.2300	1.9116	1.6121

Experiment II. Here we will consider the case: u satisfies the homogeneous boundary

Table 4: Errors and ratios in time with $u=1$, $D=10^{-4}$, $x_0=0.15$, $\sigma=0.03$ and $\Delta x=\frac{1}{4000}$.

	Δt	1/40	1/50	1/60	1/70	1/80
C-FD	E_∞	4.8518e-02	3.8959e-02	3.2533e-02	2.7916e-02	2.4440e-02
	Ratio	-	0.9834	0.9887	0.9927	0.9960
	E_2	1.7886e-02	1.4364e-02	1.1996e-02	1.0295e-02	9.0134e-03
	Ratio	-	0.9826	0.9880	0.9921	0.9955
MPC-FD	E_∞	6.4134e-05	2.9339e-05	1.6937e-05	1.2176e-05	9.0166e-06
	Ratio	-	3.5048	3.0232	2.1292	2.2499
	E_2	1.5316e-05	7.9686e-06	5.0360e-06	3.5220e-06	2.6304e-06
	Ratio	-	2.9282	2.5169	2.3197	2.1861

Table 5: Errors and ratios in time with $u=1$, $D=10^{-4}$, $x_0=0.15$, $\sigma=0.03$ and $\Delta t=\frac{1}{90}$.

	Δx	1/100	1/200	1/300	1/400	1/500
C-FD	E_∞	4.6191e-02	3.7205e-02	3.1095e-02	2.6671e-02	2.3320e-02
	Ratio	-	0.9695	0.9838	0.9954	1.0054
	E_2	1.6900e-02	1.36246e-02	1.1396e-03	9.7831e-03	8.5605e-03
	Ratio	-	0.9656	0.9792	0.9902	0.9997
MPC-FD	E_∞	7.3044e-03	1.8623e-03	8.2471e-04	4.6005e-04	2.9115e-04
	Ratio	-	2.0455	2.0755	2.1125	2.1529
	E_2	1.7151e-03	2.0088e-04	1.9158e-04	2.0256e-04	2.0451e-05
	Ratio	-	2.0543	2.0918	2.0506	2.0770

Table 6: Errors and ratios in space with $u=1$, $D=10^{-4}$, $x_0=0.15$, $\sigma=0.03$ and $\Delta t=\frac{1}{100}$.

	Δx	1/100	1/200	1/300	1/400	1/500
C-FD	E_∞	9.5352e-03	4.8617e-03	3.2220e-03	2.3953e-03	1.8976e-03
	Ratio	-	0.9808	1.0146	1.0307	1.0438
	E_2	1.8639e-03	9.2256e-04	6.0876e-04	4.5186e-04	3.5773e-04
	Ratio	-	1.0146	1.0253	1.0360	1.0469
MPC-FD	E_∞	7.2825e-0	1.9315e-04	8.6757e-05	4.8968e-05	3.1380e-05
	Ratio	-	1.9147	1.939	1.9881	1.9943
	E_2	1.5579e-04	3.9465e-05	1.7803e-05	1.0302e-05	6.9057e-06
	Ratio	-	1.9810	1.9633	1.9016	1.7925

condition. We still take $\Omega=[0,1]$, $\phi(x)=1$ and $k=1$. And set $[0,T]=[0,1]$, $x_0=0.45$, $\sigma=0.10$, $g(x,t)=-e^t\sin(2\pi x)$. For the fixed space steps $\Delta x=\frac{1}{2000}$, $\Delta x=\frac{1}{4000}$ and diffusion coefficient $D=10^{-4}$, we give the L^∞ and L^2 errors results with different time sizes, see Tables 7-8. While for the fixed time increment $\Delta t=\frac{1}{350}$ and $\Delta t=\frac{1}{400}$, we give the errors and ratios with different space steps, see the Tables 9-10. These numerical results show that our method still keeps second-order accuracy in time and second-order accuracy in space.

Experiment III. In this experiment, we take $\Omega=[0,6]$, $[0,T]=[0,2]$, $x_0=0.9$, $k=1$, $\sigma=0.08$,

Table 7: Errors and ratios in time with $g(x,t)=-e^t\sin(2\pi x)$, $D=10^{-4}$, $x_0=0.45$, $\sigma=0.10$ and $\Delta x=\frac{1}{2000}$.

	Δt	1/30	1/40	1/50	1/60	1/70
C-FD	E_∞	3.2823e-02	2.4497e-02	1.9498e-02	1.6165e-02	1.3785e-02
	Ratio	-	1.0175	1.0228	1.0281	1.0334
	E_2	1.3821e-02	1.0314e-02	8.2092e-03	6.8060e-03	5.8037e-03
	Ratio	-	1.0177	1.0229	1.0282	1.0335
MPC-FD	E_∞	2.8882e-04	1.3829e-04	8.6339e-05	5.9873e-05	4.4947e-05
	Ratio	-	2.5598	2.1112	2.0078	1.8602
	E_2	1.8639e-04	9.8895e-05	6.4686e-05	4.6176e-05	3.4405e-05
	Ratio	-	2.2030	1.9024	1.8489	1.9088

Table 8: Errors and ratios in time with $g(x,t)=-e^t\sin(2\pi x)$, $D=10^{-4}$, $x_0=0.45$, $\sigma=0.10$ and $\Delta x=\frac{1}{4000}$.

	Δt	1/30	1/40	1/50	1/60	1/70
C-FD	E_∞	3.3.78e-02	2.4747e-02	1.9748e-02	1.6415e-02	1.4035e-02
	Ratio	-	1.0086	1.0112	1.0138	1.0164
	E_2	1.3927e-02	1.0419e-02	8.3144-03	6.9112e-03	5.9089e-03
	Ratio	-	1.0087	1.0113	1.0139	1.0164
MPC-FD	E_∞	2.8433e-04	1.3538e-04	8.4173e-05	5.7913e-05	4.2858e-05
	Ratio	-	2.5792	2.1298	2.0510	1.9529
	E_2	1.8321e-04	9.6391e-05	6.2724e-05	4.4345e-05	3.2498e-05
	Ratio	-	2.2324	1.9256	1.9018	2.0164

Table 9: Errors and ratios in space with $g(x,t)=-e^t\sin(2\pi x)$, $D=10^{-4}$, $x_0=0.45$, $\sigma=0.10$ and $\Delta t=\frac{1}{350}$.

	Δx	1/200	1/220	1/240	1/260	1/280
C-FD	E_∞	4.8274e-02	3.8714e-02	3.2287e-02	2.7670e-02	2.4193e-02
	Ratio	-	0.9891	0.9957	1.0010	1.0051
	E_2	1.7798e-02	1.4275e-02	1.1907e-02	1.0205e-02	8.9234e-03
	Ratio	-	0.9883	0.9950	1.0004	1.0051
MPC-FD	E_∞	9.1615e-06	7.7167e-06	6.5053e-06	5.5506e-06	4.7120e-06
	Ratio	-	1.8006	1.9626	1.9828	2.2104
	E_2	5.0802e-06	4.1128e-06	3.3409e-06	2.7501e-06	2.2564e-06
	Ratio	-	2.2165	2.3888	2.4310	2.6703

Table 10: Errors and ratios in space with $g(x,t)=-e^t\sin(2\pi x)$, $D=10^{-4}$, $x_0=0.45$, $\sigma=0.10$ and $\Delta t=\frac{1}{400}$.

	Δx	1/200	1/220	1/240	1/260	1/280
C-FD	E_∞	3.2328e-02	2.3997e-02	1.8998e-02	1.5665e-02	1.3285e-02
	Ratio	-	1.0358	1.0468	1.0579	1.0693
	E_2	1.3611e-02	1.0104e-02	7.9987e-03	6.5955e-03	5.5932e-03
	Ratio	-	1.0359	1.0469	1.0580	1.0693
MPC-FD	E_∞	7.1531e-06	5.9489e-06	4.9110e-06	4.0346e-06	3.3341e-06
	Ratio	-	1.9342	2.2035	2.4558	2.5733
	E_2	3.7456e-06	2.9894e-06	2.3752e-06	1.8830e-06	1.5047e-06
	Ratio	-	2.3659	2.6433	2.9015	3.0258

Table 11: Comparison of mass Errors by C-FD and MPC-FD with $g(x,t)=0.1e^t$.

T	D	t	Δx	Δt	C-FD-QI	C-FD-LI	MPC-FD
2	0	0	$\frac{1}{120}$	$\frac{1}{20}$	4.4524e-2	4.4539e-2	5.9212e-17
2	10^{-5}	0	$\frac{1}{120}$	$\frac{1}{20}$	4.4524e-2	4.4539e-2	1.4211e-15
2	10^{-3}	0	$\frac{1}{120}$	$\frac{1}{20}$	4.4523e-2	4.4539e-2	7.4015e-16

$\phi(x)=1$ and $g(x,t)=0.1e^t$. For the fixed space step $\Delta x=\frac{1}{120}$ and $\Delta t=\frac{1}{20}$, and C-FD-LI, C-FD-QI are standard characteristic finite difference methods based on linear and quadratic interpolations as in [20]. we give the mass errors with different coefficients D in Table 11. We can see clearly that our scheme conserves the mass perfectly.

3.2 Two dimensional case

Define L^∞-norm and L^2-norm as follows:

$$E_\infty^n=\max_{i,j}\{|c(x_i,y_j,t^n)-C_{i,j}^n|\}, \tag{3.3a}$$

$$E_2^n=\sqrt{\sum_i\sum_j\Delta x\Delta y(c(x_i,,y_j,t^n)-C_{i,j}^n)^2}. \tag{3.3b}$$

We take the exact solution p and c of the miscible displacement equation, which are given as

$$p(x,y,t)=-\sin(\pi x)\sin(\pi y), \tag{3.4a}$$

$$c(x,y,t)=\frac{\sigma^2}{\sigma^2+2Dt}\exp\Big(-\frac{(x*-x_0)^2+(y*-y_0)^2}{2(\sigma^2+2Dt)}\Big), \tag{3.4b}$$

$$x*=x\cos(4t)+y\sin(4t),\quad y*=-x\sin(4t)+y\cos(4t). \tag{3.4c}$$

For simplicity, we select the domain $\Omega=[0,1]\times[0,1]$, and suppose $\phi(\mathbf{x})=1$, $b(c)=d(c)=r(c)=1$, and $\mathbf{u}=(\pi\cos(\pi x)\sin(\pi y),\pi\cos(\pi y)\sin(\pi x))$.

Experiment IV. In this experiment, set $[0,T]=[0,\pi/2]$, $x_0=0.35$, $y_0=0.35$, and $\sigma=0.09$. Define the discrete mass $Mass_h$ by

$$Mass_h=\sum_{i=1}^{I}\sum_{j=1}^{J}C_{i,j}\Delta x\Delta y.$$

For the fixed time step $\Delta t=\frac{1}{500}$, we give the L^∞, L^2 errors results and discrete mass errors with different time step and diffusion coefficients D, see Table 12. And then, taking space step

$$\Delta x=\Delta y=h=\frac{1}{400},$$

Table 12: Errors and ratios in time with $D=10^{-5}$, $x_0=0.35$, $y_0=0.35$,$\sigma=0.09$ and $\Delta t=\frac{1}{500}$.

	h	1/320	1/330	1/340	1/350	1/360
	E_∞	5.6470e-03	5.52158e-03	5.3847e-03	5.2508e-03	5.1065e-03
	Ratio	-	0.7301	0.8408	0.8685	0.9893
C-FD	E_2	1.5513e-03	1.5163e-03	1.4808e-03	1.4447e-03	1.4082e-03
	Ratio	-	0.7408	0.79488	0.85078	0.9082
	E_{mass}	4.9748e-04	4.8235e-04	4.6810e-04	4.5466e-04	4.4197e-04
	E_∞	2.3705e-03	2.2417e-03	2.1056e-03	1.9687e-03	1.8251e-03
	Ratio	-	1.8164	2.0973	2.3192	2.6883
MPC-FD	E_2	3.6154e-05	3.3761e-05	3.1473e-05	2.9304e-05	2.7276e-05
	Ratio	-	2.2253	2.3510	2.4626	2.5459
	E_{mass}	1.1890e-17	1.1374e-17	1.00900e-17	1.0462e-17	1.0056e-17

Table 13: Errors and ratios in time with $D=10^{-5}$, $x_0=0.35$, $y_0=0.35$, $\sigma=0.09$ and $h=\frac{1}{400}$.

	Δt	$\pi/720$	$\pi/740$	$\pi/760$	$\pi/780$	$\pi/800$
	E_∞	1.6653e-02	1.6110e-02	1.5595e-02	1.5105e-02	1.4640e-02
	Ratio	-	1.2108	1.2191	1.2276	1.2361
C-FD	E_2	5.9343e-03	5.7524e-03	5.5799e-03	5.4160e-03	5.2603e-03
	Ratio	-	1.1366	1.1419	1.1472	1.1524
	E_{mass}	3.9062e-04	3.9084e-04	3.9104e-04	3.9124e-04	3.9142e-04
	E_∞	2.6625e-03	2.5255e-03	2.4019e-03	2.2901e-03	2.1883e-03
	Ratio	-	1.9274	1.8824	1.8349	1.7952
MPC-FD	E_2	6.0722e-05	5.7784e-05	5.5020e-05	5.2417e-05	4.9965e-05
	Ratio	-	1.8096	1.8380	1.8658	1.8926
	E_{mass}	8.6501e-18	8.6523e-18	8.6544e-18	8.6564e-18	8.6582e-18

we give the errors, ratios and discrete mass errors with different time steps, see Table 13. These numerical results suggest that the mass-preserving characteristic finite difference method keeps second-order accuracy both in time and space for two dimensional problem, and preserves mass better than C-FD method.

4 Conclusions

In this paper, combining the characteristic technique with mass-preserving interpolation, we propose a new mass-preserving characteristic finite difference method for incompressible miscible displacement problem in porous media. The new scheme not only keeps mass balance but also is of the time second-order accuracy and the space high order accuracy. To illustrate our method, we consider one dimensional model problem. While we present some numerical results for both one and two dimensional problems to show the convergence and the accuracy. In fact, we can easily extend this method to more complicated multi-dimensional nonlinear coupled problems, which is our future work.

In order to keep mass balance and high accuracy both in time and space, we have used more complicated interpolation approximations in our method, which brings so many troubles for theoretical analysis that we can not give the stability and error analysis until now. In the future, we will continue to pay attention to the progress of this field and expect to give the corresponding theoretical analysis.

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