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UNIFORM SUPERCONVERGENCE ANALYSIS OF A TWO-GRID MIXED FINITE ELEMENT METHOD FOR THE TIME-DEPENDENT BI-WAVE PROBLEM MODELING D-WAVE SUPERCONDUCTORS*

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Abstract

In this paper, a two-grid mixed finite element method (MFEM) of implicit Backward Euler (BE) formula is presented for the fourth order time-dependent singularly perturbed Bi-wave problem for d-wave superconductors by the nonconforming EQ_1^{rot} element. In this approach, the original nonlinear system is solved on the coarse mesh through the Newton iteration method, and then the linear system is computed on the fine mesh with Taylor's expansion. Based on the high accuracy results of the chosen element, the uniform superclose and superconvergent estimates in the broken H^1- norm are derived, which are independent of the negative powers of the perturbation parameter appeared in the considered problem. Numerical results illustrate that the computing cost of the proposed two-grid method is much less than that of the conventional Galerkin MFEM without loss of accuracy.

Mathematics subject classification: 65M60, 65N12, 65N30.
Key words: Time-dependent Bi-wave problem, Two-grid mixed finite element method, Uniform superclose and superconvergent estimates.

1. Introduction

Superconductors are materials that have no resistance to the electric current at a T_c (critical temperature) [1]. In the state of low-T_c superconductivity, electrons are found to pair in a form and move together in a spherical orbit but in the opposite direction, which is often called s-wave [2] and Ginzburg-Landau-type models were generally proposed to describe this phenomenon [3]. For high-T_c superconductivity, electrons have been strongly proved to travel together in orbits as a four-leaf clover for d-wave pairing symmetry [4, 5] and researchers have studied various generalizations of Ginzburg-Landau-type models to explain high-T_c superconductors [6].

In the time-dependent versions of Ginzburg-Landau-type for d-wave superconductor [7-9], there exist two scalar order parameters ψ_s and ψ_d whose magnitudes represent the density of superconducting charge carriers and the parameter $\delta = -\frac{1}{\beta}$, where β is related to the ratio $\frac{ln(T_{s0}/T)}{ln(T_{d0}/T)}$ with T_{s0} and T_{d0} being the critical temperatures of s-wave and d-wave components. Some studies have shown that when $\beta \to -\infty$, s-wave component diminished and d-wave component became the leading term [8], and superconductor will be completely d-wave as

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$T \to T_{d0}(T_{s0} < T_{d0})$ [9]. Therefore, the following fourth order time-dependent singularly perturbed Bi-wave problem emerged from the time-dependent Ginzburg-Landau-type model in the case $\beta \to -\infty$:

$$\begin{cases} \psi_t + \delta\theta^2\psi - \Delta\psi + f(\psi) = 0, & (X,t) \in \Omega \times J, \\ \psi = \frac{\partial\psi}{\partial\bar{n}} = 0, & (X,t) \in \partial\Omega \times J, \\ \psi(X,0) = \psi_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

where $X = (x,y)$, $J = (0,T]$, $\psi_t = \frac{\partial\psi}{\partial t}$, θ is the bi-wave operator,

$$\theta\psi = \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}, \quad \theta^2\psi = \frac{\partial^4\psi}{\partial x^4} - 2\frac{\partial^4\psi}{\partial x^2 y^2} + \frac{\partial^4\psi}{\partial y^4}, \quad \bar{n} = (n_1, -n_2), \quad \frac{\partial\psi}{\partial\bar{n}} = \nabla\psi \cdot \bar{n}.$$

$\Omega \subset R^2$ is a bounded domain with the boundary $\partial\Omega$, and $n = (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$. $f(\psi) = \psi^3 - \psi$ and $\psi_0(X)$ is a known smooth function. Hence, $0 < \delta < 1$ is expected to be small for d-wave superconductors and problem (1.1) degenerates into the semilinear parabolic equation when $\delta \to 0$.

In recent years, there are some theoretical analysis and numerical simulations about FEMs, such as optimal order error estimates of conforming Galerkin FEMs and the modified Morley-type discontinuous Galerkin FEMs in [10, 11], uniform superconvergence error estimates of Ciarlet-Raviart schemes with the conforming and nonconforming elements in [12–14]. But these work mainly focused on the stationary singularly perturbed Bi-wave problems. Thus, to develop an effective computational method for investigating problem (1.1) is of more practical significance. As a highly efficient and accurate method, the two-grid method was proposed by [15, 16] for the nonsymmetric and nonlinear problems and has been well applied to deal with many types of problems for optimal or superconvergent error estimates, such as parabolic equation [17, 18], hyperbolic problem [19], Ginzburg-Landau equation [20], Benjamin-Bona-Mahony equation [21], and so on. Nevertheless, no studies on uniform superconvergence analysis of two-gird MFEM for problem (1.1) exists in the literature and whether the error estimate result will be relevant to the negative powers of the perturbation parameter δ or not still remains open.

In this paper, as a first attempt, we will formulate a two-grid efficient algorithm of MFEM for problem (1.1) by the nonconforming EQ_1^{rot} element and analyze the corresponding uniform superclose and superconvergence behavior, which is independent of the negative powers of the parameter. The main reasons for the uniform error estimates are as follows: the first is that the suited approximation scheme is developed, the second is that the special characters of the chosen element are employed (see the formulas (2.1)-(2.3) below), the third is that the equation includes the positive term $-\Delta$. The reminder of the paper is organized as follows. In Section 2, the stability of the numerical solution is proved and uniform superconvergence result of the semi-discrete scheme for problem (1.1) is derived. In Section 3, the superclose estimates of order $O(H^2 + \tau)$ and order $O(H^4 + h^2 + \tau)$ for the two-grid method are demonstrated, respectively, where H and h are the subdivision parameters on the coarse and fine meshes, and τ , the time step. Moreover, the corresponding global superconvergence result of order $O(H^4 + h^2 + \tau)$ is obtained through the interpolated postprocessing approach. It should be mentioned that the error estimates obtained herein is independent of the negative powers of the perturbation parameter δ by use of the high accuracy characters of the selected element. In the last section, some numerical results are conducted to confirm the theoretical analysis and indicate that the computing cost of the proposed two-grid method is less than a half of the traditional Galerkin MFEM.

Throughout this paper, we denote the $L^2(\Omega)$ inner product $(\cdot,\cdot)$ with the norm $\|\cdot\|_0$, and let $H_0^1(\Omega)=\{v\in H^1(\Omega): v|_{\partial\Omega}=0\}$. Further, we quote the classical Sobolev spaces $W^{m,p}(\Omega)$, $1\le p\le\infty$, with norm $||\cdot||_{m,p}$. When $p=2$, we simply rewrite $||\cdot||_{m,p}$ as $||\cdot||_m$. Besides, we define the space $L^p(\bar{J};Y)$ with the norm $\|f\|_{L^p(\bar{J};Y)}=(\int_0^T\|f(\cdot,t)\|_Y^p dt)^{\frac{1}{p}}$, and if $p=\infty$, the integral is replaced by the essential supremum.

2. Uniform Superconvergence Analysis for Semi-discrete Scheme

To begin with, we introduce the following space:

$$V=\left\{v\in H_0^1(\Omega),\ \theta v\in L^2(\Omega),\ \frac{\partial v}{\partial\bar{n}}|_{\partial\Omega}=0\right\}.$$

Let $\mathcal{T}_h$ be a regular rectangular partition of Ω with mesh size $h\in(0,1)$. For $T\in\mathcal{T}_h$, let its four vertices and edges be a_i and $l_i=\overline{a_ia_{i+1}}$ $(i=1,2,3,4.mod(4))$, respectively. Then the nonconforming EQ_1^{rot} element space V_{h0} is defined as [22]:

$$V_{h0}=\left\{v_h: v_h|_T\in span\{1,x,y,x^2,y^2\},\ \int_l[v_h]ds=0, l\subset\partial T,\forall\ T\in\mathcal{T}_h\right\},$$

where $[v_h]$ refers to the jump of v_h across the internal edge l, and it is v_h equals to itself if l is $\partial\Omega$. The associated interpolation operator $I_h|_T=I_T$ on V_{h0} is defined by

$$\int_{l_i}(I_h\nu-\nu)ds=0,\quad i=1\sim 4,\quad \int_T(I_h\nu-\nu)dxdy=0.$$

Then, for $w_h\in V_{h0}$, $\nu\in H^4(\Omega)$, the following conclusions, which are useful to our uniform superconvergent analysis, have been proven in [23–26], respectively.

$$(i)\quad \sum_T\int_T\frac{\partial(\nu-I_h\nu)}{\partial x}\frac{\partial w_h}{\partial x}dxdy=\sum_T\int_T\frac{\partial(\nu-I_h\nu)}{\partial y}\frac{\partial w_h}{\partial y}dxdy=0, \tag{2.1}$$

$$(ii)\quad \sum_T\int_{\partial T}\frac{\partial\nu}{\partial x}n_xw_hds=\sum_T\int_{\partial T}\frac{\partial\nu}{\partial y}n_yw_hds=\left\{\begin{array}{l}O(h^2)\|\nu\|_3\|w_h\|_h,\\ O(h^2)\|\nu\|_4\|w_h\|_0,\end{array}\right. \tag{2.2}$$

$$(iii)\quad \|w_h\|_{0,2m}\le C\|w_h\|_h,\qquad m=1,2,\ldots. \tag{2.3}$$

Here and later, $\|\cdot\|_h=(\sum_T|\cdot|_{1,T}^2)^{\frac{1}{2}}$ is a norm on V_{h0}, C denotes a positive constant irrelevant to H, h, τ and δ.

The following theorem concerns the well-posedness of problem (1.1), and its proof can be found in [27].

Theorem 2.1. *Let $\psi_0(X)\in V$, for $T>0$, problem* (1.1) *has a unique weak solution ψ on $\bar{J}$, such that ψ satisfies*

$$\psi\in L^\infty(\bar{J};V),\quad \psi_t\in L^2(\bar{J};L^2(\Omega)).$$

Now, we set $\phi=\sqrt{\delta}\theta\psi$ and consider the weak formulation of (1.1): find $\{\psi,\phi\}\in H_0^1(\Omega)\times H^1(\Omega)$, such that

$$\left\{\begin{array}{ll}(\psi_t,\nu)-\sqrt{\delta}(\bar{\nabla}\phi,\nabla\nu)+(\nabla\psi,\nabla\nu)+(f(\psi),\nu)=0, & \forall\nu\in H_0^1(\Omega),\quad t\in J,\\ (\phi,\omega)+\sqrt{\delta}(\bar{\nabla}\psi,\nabla\omega)=0, & \forall\omega\in H^1(\Omega),\quad t\in J,\\ \psi(X,0)=\psi_0(X), & X\in\Omega,\end{array}\right. \tag{2.4}$$

where $\bar{\nabla}\nu = (\frac{\partial \nu}{\partial x}, -\frac{\partial \nu}{\partial y})$.

The semi-discrete approximation scheme to (2.4) is to seek $\{\psi_h, \phi_h\} \in V_{h0} \times V_h$, such that

$$\begin{cases} (\psi_{ht}, \nu_h) - \sqrt{\delta}(\bar{\nabla}_h \phi_h, \nabla_h \nu_h) + (\nabla_h \psi_h, \nabla_h \nu_h) + (f(\psi_h), \nu_h) = 0, & \forall \nu_h \in V_{h0}, \quad t \in J, \\ (\phi_h, \omega_h) + \sqrt{\delta}(\bar{\nabla}_h \psi_h, \nabla_h \omega_h) = 0, & \forall\ \omega_h \in V_h, \quad t \in J, \\ \psi_h(0) = I_h \psi_0(X), \end{cases} \tag{2.5}$$

where ∇_h or $\bar{\nabla}_h$ denotes the gradient operators piecewisely, $(\hat{\nabla}_h *, \hat{\nabla}_h \star)_h = \sum_T (\hat{\nabla} *, \hat{\nabla} \star)_T$, $(\hat{\nabla} = \nabla$ or $\bar{\nabla})$.

Since V_{h0} is a finite dimensional space, problem (2.5) may be written as a system of ordinary differential algebraic equations. Applying Picard's theorem, it follows easily that the system has a unique solution locally for $[0, t)$. In order to prove the global existence in $[0, T]$, we need a priori bound. Now we will turn to analyze the stability of the discrete solution of (2.5).

Lemma 2.1. *With* $\{\psi_h, \phi_h\}$ *defined in* (2.5), *we have*

$$\|\psi_h(t)\|_h + \|\phi_h(t)\|_0 \le \|\psi_h(0)\|_h + \|\phi_h(0)\|_0. \tag{2.6}$$

Proof.

Differentiating the second equation of (2.5) with respect to t gives

$$(\phi_{ht}, \omega_h) + \sqrt{\delta}(\bar{\nabla}_h \psi_{ht}, \nabla_h \omega_h) = 0. \tag{2.7}$$

With $\nu_h = \psi_{ht}$ in the first equation of (2.5) and $\omega_h = \phi_h$ in (2.7), then adding together, we have

$$\|\psi_{ht}\|_0^2 + \frac{1}{2}\frac{d}{dt}(\|\psi_h\|_h^2 + \|\phi_h\|_0^2) = -(f(\psi_h), \psi_{ht}). \tag{2.8}$$

Since $F(\psi_h) = \frac{1}{4}(1 - \psi_h^2)^2 \ge 0$ and $-(f(\psi_h), \psi_{ht}) = -\frac{d}{dt}(F(\psi_h), 1)$, we derive $-(f(\psi_h), \psi_{ht}) \le 0$. Noting that $\|\psi_{ht}\|_0^2 \ge 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\psi_h\|_1^2 + \|\phi_h\|_0^2) \le 0.$$

After integration with respect to t, this shows

$$\|\psi_h(t)\|_h + \|\phi_h(t)\|_0 \le \|\psi_h(0)\|_h + \|\phi_h(0)\|_0,$$

which completes the proof. □

We are now ready for the error estimate of the semi-discrete scheme.

Theorem 2.2. *Let* $\{\psi, \phi\}$ *and* $\{\psi_h, \phi_h\}$ *be the solutions of* (2.4) *and* (2.5), *respectively. Assume that* $\psi_t \in H^2(\Omega)$, $\phi, \phi_t \in H^3(\Omega)$, $\psi \in H^4(\Omega)$, *we have*

$$\begin{aligned} &\|I_h\psi - \psi_h\|_h + \|I_h\phi - \phi_h\|_0 \\ &\le Ch^2(\|\psi_t\|_2 + \sqrt{\delta}\|\phi\|_3 + \sqrt{\delta}\|\phi_t\|_3 + \sqrt{\delta}\|\psi\|_4), \qquad \text{for } t \in \bar{J}. \end{aligned} \tag{2.9}$$

Proof. Let $\psi - \psi_h = (\psi - I_h\psi) + (I_h\psi - \psi_h) \triangleq \eta + \xi$, $\phi - \phi_h = (\phi - I_h\phi) + (I_h\phi - \phi_h) \triangleq \rho + \vartheta$. Then from (2.4) and (2.5), there hold the following error equations:

$$\begin{cases} (\xi_t, \nu_h) - \sqrt{\delta}(\bar{\nabla}_h \vartheta, \nabla_h \nu_h) + (\nabla_h \xi, \nabla_h \nu_h) = -(\eta_t, \nu_h) + \sqrt{\delta}(\bar{\nabla}_h \rho, \nabla_h \nu_h) - \sqrt{\delta}\sum\limits_T \int_{\partial T} \bar{\nabla}\phi \cdot \vec{n}\nu_h ds \\ -(\nabla_h \eta, \nabla_h \nu_h) + \sum\limits_T \int_{\partial T} \nabla\psi \cdot \vec{n}\nu_h ds + (f(\psi) - f(\psi_h), \nu_h), \\ (\vartheta, \omega_h) + \sqrt{\delta}(\bar{\nabla}_h \xi, \nabla_h \omega_h) = -(\rho, \omega_h) - \sqrt{\delta}(\bar{\nabla}_h \eta, \nabla_h \omega_h) + \sqrt{\delta}\sum\limits_T \int_{\partial T} \bar{\nabla}\psi \cdot \vec{n}\omega_h ds. \end{cases} \tag{2.10}$$

Differentiating the second equation of (2.10) with respect to t shows

$$(\vartheta_t,\omega_h)+\sqrt{\delta}(\bar{\nabla}_h\xi_t,\nabla_h\omega_h)=-(\rho_t,\omega_h)-\sqrt{\delta}(\bar{\nabla}_h\eta_t,\nabla_h\omega_h). \tag{2.11}$$

Setting $\nu_h=\xi_t$ in (2.10) and $\omega_h=\vartheta$ in (2.11), we find that

$$\begin{aligned}\|\xi_t\|_0^2+\frac{1}{2}\frac{d}{dt}(\|\xi\|_h^2+\|\vartheta\|_0^2)&=-(\eta_t,\xi_t)+\sqrt{\delta}(\bar{\nabla}_h\rho,\nabla_h\xi_t)-\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi\cdot\vec{n}\xi_t ds\\ &-(\nabla_h\eta,\nabla_h\xi_t)+\sum_T\int_{\partial T}\nabla\psi\cdot\vec{n}\xi_t ds+(f(\psi)-f(\psi_h),\xi_t)-(\rho_t,\vartheta)-\sqrt{\delta}(\bar{\nabla}_h\eta_t,\nabla_h\vartheta)\\ &+\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\psi\cdot\vec{n}\vartheta ds=:\sum_{i=1}^{9}A_i.\end{aligned}$$

Here, we find at once

$$\begin{aligned}&A_1+A_5\le Ch^2\|\psi_t\|_2\|\xi_t\|_0+Ch^2\|\psi\|_4\|\xi_t\|_0\le Ch^4(\|\psi_t\|_2^2+\|\psi\|_4^2)+\frac{1}{2}\|\xi_t\|_0^2,\\ &A_2+A_4+A_8=0,\\ &A_7+A_9\le Ch^2\|\phi_t\|_2\|\vartheta\|_0+C\sqrt{\delta}h^2\|\psi\|_4\|\vartheta\|_0\le Ch^4(\|\phi_t\|_2^2+\delta\|\psi\|_4^2)+C\|\vartheta\|_0^2.\end{aligned}$$

Using the formula $(a,b_t)=\frac{d}{dt}(a,b)-(a_t,b)$, we conclude that

$$\begin{aligned}A_3&=\frac{d}{dt}\left(\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi\cdot\vec{n}\xi ds\right)-\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi_t\cdot\vec{n}\xi ds\\ &\le|\frac{d}{dt}\left(\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi\cdot\vec{n}\xi ds\right)|+C\delta h^4\|\phi_t\|_3^2+C\|\xi\|_h^2.\end{aligned}$$

By the estimate of $\|\psi_h\|_h$ shown in Lemma 2.1 and formula 2.3, there holds

$$\begin{aligned}A_6&\le C\|\psi-\psi_h\|_{0,6}(\|\psi\|_{0,6}^2+\|\psi\|_{0,6}\|\psi_h\|_{0,6}+\|\psi_h\|_{0,6}^2)\|\xi_t\|_0+C\|\psi-\psi_h\|_0\|\xi_t\|_0\\ &\le C\|\psi-\psi_h\|_{0,6}\|\xi_t\|_0+C\|\psi-\psi_h\|_0\|\xi_t\|_0\\ &\le C(h^2(\|\psi\|_{2,6}+\|\psi\|_2)+\|\xi\|_h)\|\xi_t\|_0,\end{aligned}$$

and hence, using Young's inequality, we arrive at

$$A_6\le Ch^4\|\psi\|_{2,6}^2+C\|\xi\|_h^2+\tfrac{1}{2}\|\xi_t\|_0^2.$$

It follows as above that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\xi\|_h^2+\|\vartheta\|_0^2)&\le Ch^4(\|\psi_t\|_2^2+\delta\|\phi_t\|_3^2+\delta\|\psi\|_4^2)+C(\|\xi\|_h^2+\|\vartheta\|_0^2)\\ &+|\frac{d}{dt}(\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi\cdot\vec{n}\xi ds)|+\left|\frac{d}{dt}\left(\sum_T\int_{\partial T}\nabla\psi\cdot\vec{n}\xi ds\right)\right|.\end{aligned}$$

We now integrate with respect to t to obtain

$$\begin{aligned}\|\xi\|_h^2+\|\vartheta\|_0^2&\le Ch^4(\|\psi_t\|_2^2+\delta\|\phi_t\|_3^2+\delta\|\psi\|_4^2)+C\int_0^t(\|\xi\|_h^2+\|\vartheta\|_0^2)ds\\ &+Ch^2(\sqrt{\delta}\|\phi\|_3+\|\psi\|_3)\|\xi\|_h,\end{aligned}$$

and hence from Gronwall's lemma, this shows

$$\|\xi\|_h + \|\vartheta\|_0 \le Ch^2(\|\psi_t\|_2 + \sqrt{\delta}\|\phi\|_3 + \sqrt{\delta}\|\phi_t\|_3 + \sqrt{\delta}\|\psi\|_4).$$

The proof is now complete. □

We now turn our attention to the global superconvergence and apply the post-processing operator I_{2h} constructed in [23] to have the following theorem.

Theorem 2.3. *Under the conditions of Theorem* 2.3*, we have*

$$\|\psi - I_{2h}\psi_h\|_h \le Ch^2(\|\psi_t\|_2 + \sqrt{\delta}\|\phi\|_3 + \sqrt{\delta}\|\phi_t\|_3 + \sqrt{\delta}\|\psi\|_4), \qquad for\ t \in \bar{J}. \tag{2.12}$$

Remark 2.1. Obviously, if we don't utilize the formula $(a, b_t) = \frac{d}{dt}(a, b) - (a_t, b)$, the requirement of ϕ in the estimates of A_3 will belong to $H^4(\Omega)$ instead of $H^3(\Omega)$ in Theorem 2.3. This is the main reason why we use the derivative transfer technique in our analysis.

3. Uniform Superconvergence Analysis for Two-grid Method

Let $\{t_n : t_n = n\tau; 0 \le n \le N\}$ be a uniform partition in time with the time step $\tau = T/N$, $\psi^n = \psi(X, t_n)$ and ψ_h^n be the approximation of ψ^n in V_{h0}. For a sequence of functions $\{\varphi^n\}_n^N$, we define $\bar{\partial}_t\varphi^n = \frac{\varphi^n - \varphi^{n-1}}{\tau}$.

We now turn to the full-discrete approximation scheme of (2.4), in which the two-grid MFEM is used in the space discretization and the backward Euler scheme is applied in the time discretization. It can be divided into the following two steps.

Step one: On the coarse grid $\mathcal{T}_H$, we solve $\{\psi_H^n, \phi_H^n\} : \bar{J} \to V_{H0} \times V_H$ for the following nonlinear system, for $\{\nu_H, \omega_H\} \in V_{H0} \times V_H$, such that

$$\begin{cases} (\bar{\partial}_t\psi_H^n, \nu_H) - \sqrt{\delta}(\bar{\nabla}_H\phi_H^n, \nabla_H\nu_H) + (\nabla_H\psi_H^n, \nabla_H\nu_H) + (f(\psi_H^n), \nu_H) = 0, \\ (\phi_H^n, \omega_H) + \sqrt{\delta}(\bar{\nabla}_H\psi_H^n, \nabla_H\omega_H) = 0, \\ \psi_H^0 = I_H\psi_0. \end{cases} \tag{3.1}$$

Step two: On the fine grid $\mathcal{T}_h$, we compute $\{\Psi_h^n, \Phi_h^n\} \in V_{h0} \times V_h$ for the following linear system, for $\{\nu_h, \omega_h\} \in V_{h0} \times V_h$, such that

$$\begin{cases} (\bar{\partial}_t\Psi_h^n, \nu_h) - \sqrt{\delta}(\bar{\nabla}_h\Phi_h^n, \nabla_h\nu_h) + (\nabla_h\Psi_h^n, \nabla_h\nu_h) + (f(\psi_H^n) + f'(\psi_H^n)(\Psi_h^n - \psi_H^n), \nu_h) = 0, \\ (\Phi_h^n, \omega_h) + \sqrt{\delta}(\bar{\nabla}_h\Psi_h^n, \nabla_h\omega_h) = 0, \\ \Psi_h^0 = I_h\Psi_0. \end{cases} \tag{3.2}$$

In what follows, we begin with the stability of the approximation solution $\{\psi_H^n, \phi_H^n\}$ to (3.1).

Lemma 3.1. *The solution* $\{\psi_H^n, \phi_H^n\}$ *to* (3.1) *satisfies*

$$\|\psi_H^n\|_h + \|\phi_H^n\|_0 \le C(\|\psi_H^0\|_h + \|\phi_H^0\|_0), \qquad 1 \le n \le N. \tag{3.3}$$

Proof. Differentiating the second equation of (3.1) with respect to t yields

$$(\bar{\partial}_t\phi_H^n, \omega_H) + \sqrt{\delta}(\bar{\nabla}_H\bar{\partial}_t\psi_H^n, \nabla_H\omega_H) = 0. \tag{3.4}$$

Taking $\nu_H = \bar{\partial}_t\psi_H^n$ in the first equation of (3.1) and $\omega_H = \phi_H^n$ in (3.4), adding together gives

$$\|\bar{\partial}_t\phi_H^n\|_0^2 + \frac{1}{2}\bar{\partial}_t(\|\nabla_H\psi_H^n\|_0^2 + \|\phi_H^n\|_0^2) + (f(\psi_H^n), \bar{\partial}_t\psi_H^n) \le 0.$$

By Taylor expansion, we have

$$(F(\psi_H^n)-F(\psi_H^{n-1}),1)=(f(\psi_H^n),\psi_H^n-\psi_H^{n-1})-(\tfrac{F''(\zeta)}{2}(\psi_H^n-\psi_H^{n-1})^2,1),$$

where $\zeta=\psi_H^n+(1-c)\psi_H^{n-1}(0<c<1)$.
Here

$$-(\tfrac{F''(\zeta)}{2}(\psi_H^n-\psi_H^{n-1})^2,1)\le\tfrac{1}{2}\|\psi_H^n-\psi_H^{n-1}\|_0^2=\tfrac{\tau^2}{2}\|\bar\partial_t\psi_H^n\|_0^2,$$

so that

$$\|\bar\partial_t\phi_H^n\|_0^2+\tfrac{1}{2}\bar\partial_t(\|\nabla\psi_H^n\|_0^2+\|\phi_H^n\|_0^2)+\tfrac{1}{\tau}(F(\psi_H^n)-F(\psi_H^{n-1}),1)\le\tfrac{\tau}{2}\|\bar\partial_t\psi_H^n\|_0^2.$$

Multiplying it by 2τ and summing it from 1 to m, we find that

$$(1-C\tau)\|\bar\partial_t\phi_H^m\|_0^2+\|\nabla\psi_H^m\|_0^2+\|\phi_H^m\|_0^2+2(F(\psi_H^m),1)\le\|\nabla\psi_H^0\|_0^2+\|\phi_H^0\|_0^2+2(F(\psi_H^0),1),$$

for a small τ, and hence

$$\|\psi_H^m\|_h^2+\|\phi_H^m\|_0^2\le C(\|\psi_H^0\|_h^2+\|\phi_H^0\|_0^2).$$

The proof is now complete. □

Next, we show the stability of the solution $\{\Psi_h^n,\Phi_h^n\}$ to (3.2).

Lemma 3.2. *The solution* $\{\Psi_h^n,\Phi_h^n\}$ *to* (3.2) *satisfies*

$$\|\Psi_h^n\|_h+\|\Phi_h^n\|_0\le C(\|\Psi_h^0\|_h+\|\Phi_h^0\|_0),\qquad 1\le n\le N. \tag{3.5}$$

Proof. In analogy with (3.4), we write

$$\|\bar\partial_t\Psi_h^n\|_0^2+\tfrac{1}{2}\bar\partial_t(\|\nabla\Psi_h^n\|_0^2+\|\Phi_h^n\|_0^2)+(f(\psi_H^n)+f'(\psi_H^n)(\Psi_h^n-\psi_H^n),\bar\partial_t\Psi_h^n)\le 0,$$

and by (3.3) this yields

$$\|\bar\partial_t\Psi_h^n\|_0^2+\tfrac{1}{2}\bar\partial_t(\|\nabla\Psi_h^n\|_0^2+\|\Phi_h^n\|_0^2)\le C\|\psi_H^n\|_{0,6}\|\bar\partial_t\Psi_h^n\|_0+C\|\Psi_h^n\|_0\|\bar\partial_t\Psi_h^n\|_0,$$

whence,

$$\tfrac{1}{2}\bar\partial_t(\|\Psi_h^n\|_h^2+\|\Phi_h^n\|_0^2)\le C(\|\psi_H^0\|_h^2+\|\phi_H^0\|_0^2)+C\|\Psi_h^n\|_h^2.$$

Here, as before, we obtain

$$\|\Psi_h^m\|_h^2+\|\Phi_h^m\|_0^2\le C\tau\sum_{i=1}^m(\|\psi_H^0\|_h^2+\|\phi_H^0\|_0^2)+C\tau\sum_{i=1}^m\|\Psi_h^n\|_h^2+C(\|\psi_h^0\|_h^2+\|\phi_h^0\|_0^2).$$

for a small τ, it follows from discrete Gronwall's inequality that

$$\|\Psi_h^m\|_h^2+\|\Phi_h^m\|_0^2\le C(\|\psi_h^0\|_h^2+\|\phi_h^0\|_0^2),$$

which completes the proof. □

Now we derive the superclose result on the coarse mesh.

Theorem 3.1. *Let $\{\psi^n,\phi^n\}$ and $\{\psi_H^n,\phi_H^n\}$ be the solutions of (2.4) and (3.1), respectively. Assume that $\psi_{tt}\in L^\infty(\bar{J};L^2(\Omega))$, $\psi_t\in L^\infty(\bar{J};H^2(\Omega))$, $\phi,\phi_t\in L^\infty(\bar{J};H^3(\Omega))$, $\psi_t\in L^\infty(\bar{J};H^4(\Omega))$, we have*

$$\|I_H\psi^n-\psi_H^n\|_H+\|I_h\phi^n-\phi_H^n\|_0=O(H^2+\tau). \tag{3.6}$$

Proof. Let $\psi^n-\psi_h^n=(\psi^n-I_h\psi^n)+(I_h\psi^n-\psi_H^n)\triangleq\eta_1^n+\xi_1^n,\phi^n-\phi_h^n=(\phi^n-I_h\phi^n)+(I_h\phi^n-\phi_H^n)\triangleq\rho_1^n+\vartheta_1^n$.
By (2.4) and (3.1), we have the following error equations:

$$\begin{cases}(\bar{\partial}_t\xi_1^n,\nu_H)-\sqrt{\delta}(\bar{\nabla}_H\vartheta_1^n,\nabla_H\nu_H)+(\nabla_H\xi_1^n,\nabla\nu_H)\\ =-(\bar{\partial}_t\eta_1^n,\nu_H)+\sqrt{\delta}(\bar{\nabla}_H\rho_1^n,\nabla_H\nu_H)-\sqrt{\delta}\sum\limits_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\nu_Hds-(\nabla_H\eta_1^n,\nabla_H\nu_H)\\ +\sum\limits_T\int_{\partial T}\nabla\psi^n\cdot\vec{n}\nu_Hds-(f(\psi^n)-f(\psi_H^n),\nu_H)+(R_1^n,\nu_H),\\ (\bar{\partial}_t\vartheta_1^n,\omega_H)+\sqrt{\delta}(\bar{\nabla}_H\bar{\partial}_t\xi_1^n,\nabla_H\omega_H)\\ =-(\bar{\partial}_t\rho_1^n,\omega_H)-\sqrt{\delta}(\bar{\nabla}_H\bar{\partial}_t\eta_1^n,\nabla_H\omega_H)+\sqrt{\delta}\sum\limits_T\int_{\partial T}\bar{\nabla}\psi^n\cdot\vec{n}\omega_Hds,\end{cases} \tag{3.7}$$

where $R_1^n=\bar{\partial}_t\psi^n-\psi_t^n$ and $\|R_1^n\|_0\le C\tau\|\psi_{tt}^n\|_0$.
Setting $\{\nu_H,\omega_H\}=\{\bar{\partial}_t\xi_1^n,\vartheta_1^n\}$ in (3.1) and adding, we have

$$\begin{aligned}&\|\bar{\partial}_t\xi_1^n\|_0^2+\frac{1}{2}\bar{\partial}_t(\|\xi_1^n\|_H^2+\|\vartheta_1^n\|_0^2)\\ \le&-(\bar{\partial}_t\eta_1^n,\bar{\partial}_t\xi_1^n)+\sqrt{\delta}(\bar{\nabla}_H\rho_1^n,\nabla_H\bar{\partial}_t\xi_1^n)-\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\bar{\partial}_t\xi_1^nds\\ &-(\nabla_H\eta_1^n,\nabla_H\bar{\partial}_t\xi_1^n)+\sum_T\int_{\partial T}\nabla\psi^n\cdot\vec{n}\bar{\partial}_t\xi_1^nds-(f(\psi^n)-f(\psi_H^n),\bar{\partial}_t\xi_1^n)-(\bar{\partial}_t\rho_1^n,\vartheta_1^n)\\ &-\sqrt{\delta}(\bar{\nabla}_H\bar{\partial}_t\eta_1^n,\nabla_H\vartheta_1^n)-\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\psi^n\cdot\vec{n}\vartheta_1^nds+(R_1^n,\bar{\partial}_t\xi_1^n)\triangleq\sum_{i=1}^{10}B_i.\end{aligned} \tag{3.8}$$

Using (2.1)-(2.2) and Young's inequality, we find that

$$\begin{aligned}&B_1+B_5\le CH^2|\bar{\partial}_t\psi^n|_2\|\bar{\partial}_t\xi_1^n\|_0+C\sqrt{\delta}H^2|\psi^n|_4\|\bar{\partial}_t\xi_1^n\|_0\le CH^4(|\psi_t^n|_2^2+\delta|\psi^n|_4^2)+\frac{1}{3}\|\bar{\partial}_t\xi_1^n\|_0^2,\\ &B_7+B_9\le CH^2|\bar{\partial}_t\phi^n|_2\|\vartheta_1^n\|_0+C\sqrt{\delta}H^2|\psi^n|_4\|\vartheta_1^n\|_0\le CH^4(|\phi_t^n|_2^2+\delta|\psi^n|_4^2)+C\|\vartheta_1^n\|_0^2,\\ &B_{10}\le C\tau\|\psi_{tt}^n\|_0\|\bar{\partial}_t\xi_1^n\|_0\le C\tau^2\|\psi_{tt}^n\|_0^2+\frac{1}{3}\|\bar{\partial}_t\xi_1^n\|_0^2,\\ &B_2+B_4+B_8=0.\end{aligned}$$

By the following formula

$$(\nu^n,\bar{\partial}_t\omega^n)=\bar{\partial}_t(\nu^n,\omega^n)-(\bar{\partial}_t\nu^n,\omega^{n-1}), \tag{3.9}$$

this shows

$$\begin{aligned}B_3=&\sqrt{\delta}\bar{\partial}_t\Big(\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\xi_1^nds\Big)-\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\bar{\partial}_t\phi^n\cdot\vec{n}\xi_1^{n-1}ds\\ \le&\sqrt{\delta}\bar{\partial}_t(\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\xi_1^nds\Big)+C\delta H^4|\phi_t^n|_3^2+C\|\xi_1^{n-1}\|_H^2.\end{aligned}$$

The analogue of A_3, together with Lemma 3.1, then shows

$$B_4 \le CH^4(|\psi^n|_{2,6}^2 + |\psi^n|_2^2) + C\|\xi_1^n\|_H^2 + \tfrac{1}{3}\|\bar{\partial}_t\xi_1^n\|_0^2.$$

Adding as above, we conclude that

$$\begin{aligned}\|\xi_1^n\|_H^2 + \|\vartheta_1^n\|_0^2 \le {} & CH^4(|\psi_t^n|_2^2 + \delta|\phi_t^n|_3^2 + \delta|\psi^n|_4^2) + C\tau^2\|\psi_{tt}^n\|_0^2 \\ & + C\tau\sum_{m=1}^{n-1}\|\xi_1^m\|_H^2 + C\tau\sum_{m=1}^{n}\|\vartheta_1^m\|_0^2 + 2\sqrt{\delta}\bar{\partial}_t\Big(\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\xi_1^n ds\Big),\end{aligned}$$

whence, for a small τ, by discrete Gronwall's lemma, this shows

$$\begin{aligned}\|\xi_1^n\|_H^2 + \|\vartheta_1^n\|_0^2 \le {} & CH^4(|\psi_t^n|_3^2 + \delta|\phi_t^n|_3^2 + \delta|\psi^n|_4^2) + C\tau^2\|\psi_{tt}^n\|_0^2 \\ & + CH^2(\sqrt{\delta}|\phi^n|_3 + |\psi^n|_3)\|\xi_1^n\|_H.\end{aligned}$$

It follows from Young's inequality that

$$\begin{aligned}\|\xi_1^n\|_H + \|\vartheta_1^n\|_0 \le {} & CH^2(|\psi^n|_{L^\infty(J;H^2(\Omega))} + \sqrt{\delta}|\phi^n|_{L^\infty(J;H^3(\Omega))} + \sqrt{\delta}|\phi_t^n|_{L^\infty(J;H^3(\Omega))} \\ & + \sqrt{\delta}|\psi^n|_{L^\infty(J;H^4(\Omega))}) + C\tau\|\psi_{tt}^n\|_{L^\infty(J;L^2(\Omega))}.\end{aligned}$$

The proof is now complete. □

Next we analyze the corresponding superclose estimate on the fine mesh.

Theorem 3.2. *Let $\{\psi^n, \phi^n\}$ and $\{\Psi_h^n, \Phi_h^n\}$ be the solutions of (2.4) and (3.2), respectively. Assume that $\psi_{tt} \in L^\infty(\bar{J}; L^\infty(\Omega))$, $\psi_t \in L^\infty(\bar{J}; H^2(\Omega))$, $\phi, \phi_t \in L^\infty(\bar{J}; H^3(\Omega))$, $\psi_t \in L^\infty(\bar{J}; H^4(\Omega))$, we have*

$$\|I_h\psi^n - \Psi_h^n\|_h + \|I_h\phi^n - \Phi_h^n\|_0 = O(h^2 + H^4 + \tau). \tag{3.10}$$

Proof. We set $\psi^n - \Psi_h^n = (\psi^n - I_h\psi^n) + (I_h\psi^n - \Psi_h^n) \triangleq \eta_2^n + \xi_2^n$, $\phi^n - \Phi_h^n = (\phi^n - I_h\phi^n) + (I_h\phi^n - \Phi_h^n) \triangleq \rho_2^n + \vartheta_2^n$. In fact, by Taylor's expansion, we have

$$f(\psi) = f(\psi_H^n) + f'(\psi_H^n)(\psi - \psi_H^n) + \tfrac{1}{2}f''(\psi_H^n + \nu(\psi - \psi_H^n))(\psi - \psi_H^n)^2, (0 \le \nu \le 1). \tag{3.11}$$

This time we have for ξ_2^n and ϑ_2^n,

$$\begin{aligned}\|\bar{\partial}_t\xi_2^n\|_0^2 + \frac{1}{2}\bar{\partial}_t(\|\nabla\xi_2^n\|_0^2 + \|\vartheta_2^n\|_0^2) \le {} & -(\bar{\partial}_t\eta_2^n, \bar{\partial}_t\xi_2^n) + \sqrt{\delta}(\bar{\nabla}_H\rho_2^n, \nabla_H\bar{\partial}_t\xi_2^n) \\ & -\sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\bar{\partial}_t\xi_2^n ds - (\nabla\eta_2^n, \nabla\bar{\partial}_t\xi_2^n) + \sum_T\int_{\partial T}\nabla\psi^n\cdot\vec{n}\bar{\partial}_t\xi_2^n ds \\ & -(f'(\psi_H^n)(\psi^n - \Psi_h^n), \bar{\partial}_t\xi_2^n) - (\frac{1}{2}f''(\psi_H^n + \nu(\psi - \psi_H^n))(\psi - \psi_H^n)^2, \bar{\partial}_t\xi_2^n) \\ & -(\bar{\partial}_t\rho_2^n, \vartheta_2^n) - \sqrt{\delta}(\bar{\nabla}_H\bar{\partial}_t\eta_2^n, \nabla_H\vartheta_2^n) - \sqrt{\delta}\sum_T\int_{\partial T}\bar{\nabla}\psi^n\cdot\vec{n}\vartheta_2^n ds + (R_1^n, \bar{\partial}_t\xi_2^n) \\ =: {} & \sum_{i=1}^{11} D_i,\end{aligned} \tag{3.12}$$

where $R_1^n = \bar{\partial}_t\psi^n - \psi_t^n$ and $\|R_1^n\|_0 \le C\tau\|\psi_{tt}^n\|_0$. Referring to the proof of Theorem 3.3, we find that

$$D_1 + D_5 + D_8 + D_{10} + D_{11} \le Ch^4(|\psi_t^n|_2^2 + |\phi_t^n|_2^2 + \delta|\psi^n|_4^2) + C\tau^2\|\psi_{tt}^n\|_0^2 + \frac{1}{3}\|\bar{\partial}_t\xi_2^n\|_0^2 + C\|\vartheta_2^n\|_0^2,$$

$$D_3 \le \sqrt{\delta}\bar{\partial}_t(\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\xi_2^n ds) + C\delta h^4|\phi_t^n|_3^2 d + C\|\xi_2^{n-1}\|_h^2,$$

$$D_2 + D_4 + D_9 = 0.$$

By the splitting technique of [13],

$$\begin{aligned} D_6 &= -(f'(\psi_H^n)(\psi^n - I_h\psi^n), \bar{\partial}_t\xi_2^n) - (f'(\psi_H^n)(I_h\psi^n - \Psi_h^n), \bar{\partial}_t\xi_2^n) \\ &= -(f'(\psi_H^n)(\psi^n - I_h\psi^n), \bar{\partial}_t\xi_2^n) - (3(\psi_H^n)^2(I_h\psi^n - \Psi_h^n), \bar{\partial}_t\xi_2^n) + (I_h\psi^n - \Psi_h^n, \bar{\partial}_t\xi_2^n), \end{aligned}$$

where now

$$-(3(\psi_H^n)^2(I_h\psi^n - \Psi_h^n), \bar{\partial}_t\xi_2^n) = -\frac{1}{\tau}(3(\psi_H^n)^2(I_h\psi^n - \Psi_h^n), I_h\psi^n - \Psi_h^n) \le 0.$$

Hence, arguing as before,

$$D_6 \le Ch^4\|\psi^n\|_{2,6}^2 + C\|\xi_2^n\|_h^2 + \frac{1}{3}\|\bar{\partial}_t\xi_2^n\|_0^2.$$

By the interpolation theory and (3.6), we obtain

$$\begin{aligned} D_7 &\le C(\|\psi^n - I_H\psi^n\|_{0,6}^2 + \|I_H\psi^n - \psi_H^n\|_{0,6}^2)\|\bar{\partial}_t\xi_2^n\|_0 \le C(H^4\|\psi^n\|_{2,6}^2 + \|I_H\psi^n - \psi_H^n\|_h^2)\|\bar{\partial}_t\xi_2^n\|_0 \\ &\le C(H^4\|\psi^n\|_{2,6}^2 + CH^4(|\psi_t^n|_2^2 + \delta|\phi^n|_3^2 + \delta|\phi_t^n|_3^2 + \delta|\psi^n|_4^2)) + C\tau^2\|\psi_{tt}^n\|_0^2)\|\bar{\partial}_t\xi_2^n\|_0 \\ &\le CH^8(\|\psi^n\|_{2,6}^2 + |\psi_t^n|_2^2 + \delta|\phi^n|_3^2 + \delta|\phi_t^n|_3^2 + \delta|\psi^n|_4^2) + C\tau^4\|\psi_{tt}^n\|_0^4 + \frac{1}{3}\|\bar{\partial}_t\xi_2^n\|_0^2. \end{aligned}$$

Altogether, this shows

$$\begin{aligned} \|\xi_2^n\|_h^2 + \|\vartheta_2^n\|_0^2 &\le C(h^4 + H^8)(|\psi_t^n|_2^2 + \delta|\phi^n|_3^2 + \delta|\phi_t^n|_3^2 + \delta|\psi^n|_4^2) + C\tau^2\|\psi_{tt}^n\|_0^2 \\ &\quad + C\tau\sum_{m=1}^{n-1}\|\xi_2^m\|_h^2 + C\tau\sum_{m=1}^{n}\|\vartheta_2^m\|_0^2 + 2\sqrt{\delta}\bar{\partial}_t\sum_T\int_{\partial T}\bar{\nabla}\phi^n\cdot\vec{n}\xi_2^{n-1}ds, \end{aligned}$$

whence, for small τ, by repeated application,

$$\begin{aligned} \|\xi_2^n\|_h^2 + \|\vartheta_2^n\|_0^2 &\le C(h^4 + H^8)(|\psi_t^n|_2^2 + \delta|\phi^n|_3^2 + \delta|\phi_t^n|_3^2 + \delta|\psi^n|_4^2) \\ &\quad + C\tau^2\|\psi_{tt}^n\|_0^2 + Ch^2\sqrt{\delta}|\phi^n|_3\|\xi_2^n\|_h, \end{aligned}$$

and hence from Young's inequality

$$\begin{aligned} \|\xi_2^n\|_h + \|\vartheta_2^n\|_0 &\le C(h^2 + H^4)(|\psi_t^n|_{L^\infty(J;H^2(\Omega))} + \sqrt{\delta}|\phi^n|_{L^\infty(J;H^3(\Omega))} \\ &\quad + \sqrt{\delta}|\phi_t^n|_{L^\infty(J;H^3(\Omega))} + \sqrt{\delta}|\psi^n|_{L^\infty(J;H^4(\Omega))}) + C\tau\|\psi_{tt}^n\|_{L^\infty(J;L^2(\Omega))}. \end{aligned}$$

The proof is now complete. □

The following global uniform superconvergence result similar to that in Theorem (2.3) also holds for fully-discrete scheme by using the interpolated postprocessing operator in above section. Firstly, we combine the adjacent four elements $T_i (i = 1, 2, 3, 4)$ into one big element $\hat{T}$. Then we define the interpolated postprocessing operator I_{2h} on $\hat{T}$ as [23]:

$$\begin{cases} I_{2h}\psi|_{\hat{T}} \in Q_2(\hat{T}), \\ I_{2h}\psi(a_i) = \psi(a_i), \quad i = 1, \cdots, 9, \end{cases}$$

which satisfies

$$\begin{cases} I_{2h}I_h\psi = I_{2h}\psi, & \\ \|I_{2h}\psi - \psi\|_1 \le Ch^2\|\psi\|_3, & \psi \in H^3(\Omega), \\ \|I_{2h}\psi\|_1 \le C\|\psi\|_1, & \forall\psi \in S_2^h, \end{cases} \tag{3.13}$$

where Q_2 is a biquadratic polynomial space, a_i are nine vertexes of $\hat{T}$, S_2^h is the biquadratic finite element space.

Then we show the following global uniform superconvergence result, which is independent of the negative powers of the parameter.

Theorem 3.3. *Under the conditions of Theorem* 3.4*, we have*

$$\|\psi^n - I_{2h}\Psi_h^n\|_1 = O(h^2 + H^4 + \tau).$$

Remark 3.1. It is not difficult to check that Theorems 2.3 and 3.3 are also valid to quasi-Wilson element [28] on rectangular mesh and modified quasi-Wilson element [29] on quadrilateral mesh. However, for the conforming triangular linear element [30], the nonconforming Q_1^{rot} element on square mesh [31] and the rectangular CNQ_1^{rot} element [32], whether Theorems 2.3 and 3.3 hold or not is an open issue. Therefore, EQ_1^{rot} element used in our present study is an appropriate choice.

4. Numerical Results

In this section, we carry out the following numerical experiment:

$$\begin{cases} \psi_t + \delta\theta^2\psi - \Delta\psi + \psi^3 - \psi = g, & (X,t) \in \Omega \times J, \\ \psi = \frac{\partial\psi}{\partial\bar{n}} = 0, & (X,t) \in \partial\Omega \times J, \end{cases} \tag{4.1}$$

with $\Omega = (0,1) \times (0,1)$, $T = 1$, the source function g is computed from the exact solution $\psi(x,y,t) = e^{-t}\sin^2(\pi x)\sin^2(\pi y)$.

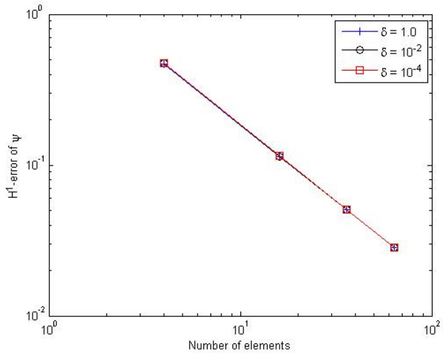
We choose $H^4 = h^2$ and $\tau = h^2$ to verify the superclose and superconvergence rates in Theorems 3.4-3.5 and use Newton iterations with 10^{-5} being the tolerance error of the iteration in our computation on the MATLAB platform. The error estimates with the rates of convergence are listed in Tables 4.1-4.6 at $t = 0.1, 1.0$ with different $\delta = 10^{-4}, \delta = 10^{-2}, 1,$

Table 4.1: Numerical results of ψ and ϕ at $t = 1.0$ with $\delta = 1.0$.

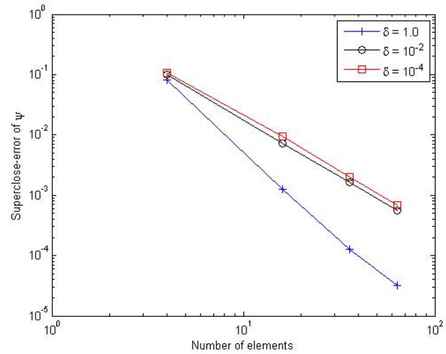
H	h	$\|\psi - \Psi_h\|_1$	*order*	$\|I_h\psi - \Psi_h\|_1$	*order*	$\|I_{2h}\Psi_h - \psi\|_1$	*order*	$\|\phi - \Phi_h\|_0$	*order*
1/2	1/4	1.9094e-01	–	3.3646e-02	–	3.4152e-01	–	7.6242e-02	–
1/4	1/16	4.6382e-02	1.0207	4.9936e-04	3.0371	2.7633e-02	1.8138	1.3740e-03	1.9480
1/6	1/36	2.0592e-02	1.0013	5.0824e-05	2.8177	5.5449e-03	1.9806	1.7627e-04	1.9875
1/8	1/64	1.1581e-02	1.0003	1.2209e-05	2.4788	1.7591e-03	1.9954	5.1646e-05	1.9976

Table 4.2: Numerical results of ψ and ϕ at $t = 0.1$ with $\delta = 1.0$.

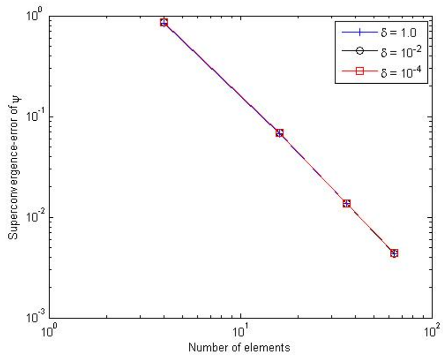
H	h	$\|\psi - \Psi_h\|_1$	*order*	$\|I_h\psi - \Psi_h\|_1$	*order*	$\|I_{2h}\Psi_h - \psi\|_1$	*order*	$\|\phi - \Phi_h\|_0$	*order*
1/2	1/4	4.6964e-01	–	8.2815e-02	–	8.4009e-01	–	1.8703e-01	–
1/4	1/16	1.1408e-01	1.0208	1.2333e-03	3.0347	6.7965e-02	1.8138	3.4137e-03	2.8879
1/6	1/36	5.0649e-02	1.0013	1.2825e-04	2.7912	1.3638e-02	1.9806	4.5510e-04	2.4848
1/8	1/64	2.8485e-02	1.0003	3.1625e-05	2.4333	4.3268e-03	1.9954	1.3604e-04	2.0988



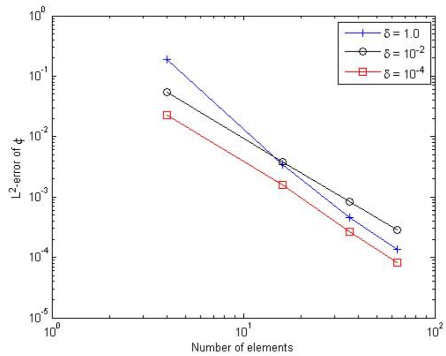
(a) Convergence error of ψ



(b) Superclose error of ψ



(c) Superconvergence error of ψ



(d) Convergence error of ϕ

Fig. 4.1. Error reductions of ψ and ϕ at $t = 0.1$ with different δ.

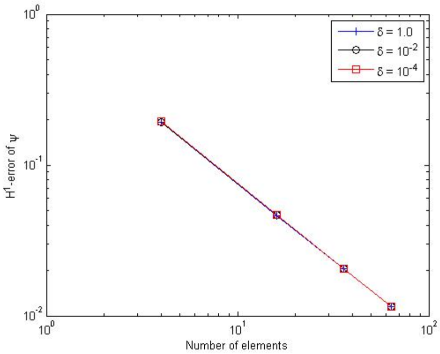
Table 4.3: Numerical results of ψ and ϕ at $t = 1.0$ with $\delta = 10^{-2}$.

H	h	$\|\psi - \Psi_h\|_1$	$order$	$\|I_h\psi - \Psi_h\|_1$	$order$	$\|I_{2h}\Psi_h - \psi\|_1$	$order$	$\|\phi - \Phi_h\|_0$	$order$
1/2	1/4	1.9267e-01	–	4.2372e-02	–	3.5039e-01	–	2.2466e-02	–
1/4	1/16	4.6473e-02	1.0258	2.9435e-03	1.9237	2.7805e-02	1.8278	1.5333e-03	1.9365
1/6	1/36	2.0602e-02	1.0031	6.2267e-04	1.9155	5.5805e-03	1.9804	3.2491e-04	1.9134
1/8	1/64	1.1583e-02	1.0008	2.1216e-04	1.8713	1.7719e-03	1.9939	1.0977e-04	1.8860

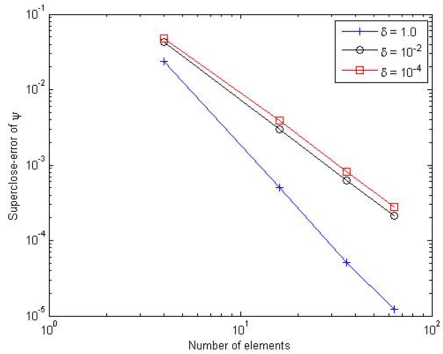
Table 4.4: Numerical results of ψ and ϕ at $t = 0.1$ with $\delta = 10^{-2}$.

H	h	$\|\psi - \Psi_h\|_1$	$order$	$\|I_h\psi - \Psi_h\|_1$	$order$	$\|I_{2h}\Psi_h - \psi\|_1$	$order$	$\|\phi - \Phi_h\|_0$	$order$
1/2	1/4	4.7307e-01	–	1.0044e-01	–	8.5988e-01	–	5.3996e-02	–
1/4	1/16	1.1430e-01	1.0246	7.2048e-03	1.9006	6.8386e-02	1.8262	3.7656e-03	1.9210
1/6	1/36	5.0675e-02	1.0030	1.6132e-03	1.8454	1.3735e-02	1.9795	8.3727e-04	1.8541
1/8	1/64	2.8491e-02	1.0008	5.5474e-04	1.8553	4.3623e-03	1.9934	2.8541e-04	1.8705

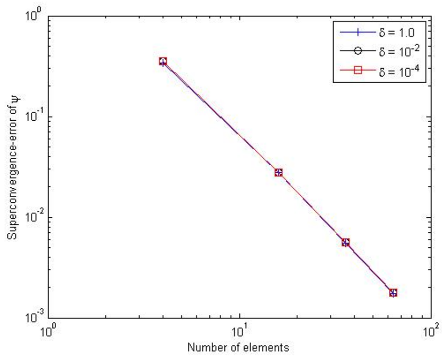
respectively. Those show that when $h \to 0$, $\|\psi - \Psi_h\|_1$ is convergent at an optimal rate of $O(h)$, $\|\Psi_h - I_h\psi\|_1$, $\|\psi - I_{2h}\Psi_h\|_1$ and $\|\phi - \Phi_h\|_0$ are convergent at a rate of $O(h^2)$, which coincides with the theoretical analysis. Meanwhile, the error reductions for ψ and ϕ at $t = 0.1, 1.0$ with different $\delta = 10^{-4} \sim 1$ are also plotted in Figs. 4.1-4.2, respectively.



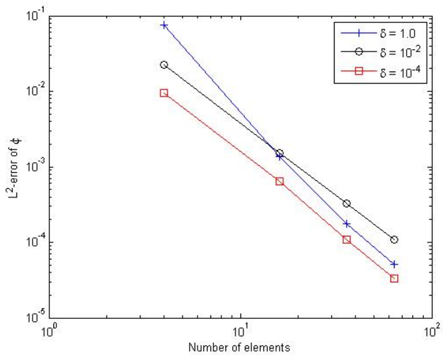
(a) Convergence error of ψ



(b) Superclose error of ψ



(c) Superconvergence error of ψ



(d) Convergence error of ϕ

Fig. 4.2. Error reductions of ψ and ϕ at $t = 1.0$ with different δ.

Table 4.5: Numerical results of ψ and ϕ at $t = 1.0$ with $\delta = 10^{-4}$.

H	h	$\|\psi - \Psi_h\|_1$	*order*	$\|I_h\psi - \Psi_h\|_1$	*order*	$\|I_{2h}\Psi_h - \psi\|_1$	*order*	$\|\phi - \Phi_h\|_0$	*order*
1/2	1/4	1.9381e-01	–	4.7315e-02	–	3.5083e-01	–	9.4671e-03	–
1/4	1/16	4.6544e-02	1.0290	3.9132e-03	1.7979	2.7936e-02	1.8253	6.4916e-04	1.9331
1/6	1/36	2.0609e-02	1.0046	8.2500e-04	1.9197	5.6070e-03	1.9803	1.0826e-04	2.2088
1/8	1/64	1.1585e-02	1.0011	2.7900e-04	1.8843	1.7812e-03	1.9931	3.2879e-05	2.0712

Table 4.6: Numerical results of ψ and ϕ at $t = 0.1$ with $\delta = 10^{-4}$.

H	h	$\|\psi - \Psi_h\|_1$	*order*	$\|I_h\psi - \Psi_h\|_1$	*order*	$\|I_{2h}\Psi_h - \psi\|_1$	*order*	$\|\phi - \Phi_h\|_0$	*order*
1/2	1/4	4.7420e-01	–	1.0563e-01	–	8.5809e-01	–	2.2947e-02	–
1/4	1/16	1.1445e-01	1.0254	9.2994e-03	1.7529	6.8663e-02	1.8218	1.6104e-03	1.9164
1/6	1/36	5.0690e-02	1.0043	2.0405e-03	1.8704	1.3793e-02	1.9793	2.6978e-04	2.2032
1/8	1/64	2.8494e-02	1.0012	6.9253e-04	1.8782	4.3821e-03	1.9929	8.1741e-05	2.0753

On the other hand, we also compare the computing cost of the conventional mixed FEM and the two-grid method (TGM) in Tables 4.7-4.11 for the same partition ($h = \frac{1}{16}$) at different time levels with different δ values. It clearly shows that the CPU time required for the latter one is only less than a half of the former one, which indicates that the proposed method herein is indeed a very effective algorithm for solving the nonlinear time-dependent Bi-wave problem.

Table 4.7: Errors and CPU cost of mixed FEM and TGM for ψ and ϕ at $t = 0.1$ with different δ.

δ	$\|I_{2h}\psi_h - \psi\|_h$	CPUs (FEM)	$\|I_{2h}\Psi_h - \psi\|_h$	CPUs (TGM)	$\|\phi_h - \phi\|_h$	CPUs (FEM)	$\|\Phi_h - \phi\|_h$	CPUs (TGM)
10^{-4}	7.2266e-02	45.73	6.8663e-02	20.90	8.0528e-03	7.24	4.0304e-03	3.61
10^{-3}	7.0720e-02	50.71	6.8590e-02	22.06	1.1964e-02	6.37	1.1884e-02	3.35
10^{-2}	6.8398e-02	47.82	6.8386e-02	24.34	1.6909e-02	6.64	3.6924e-02	3.47
10^{-1}	6.8009e-02	54.04	6.8031e-02	23.27	8.1362e-02	6.66	1.1182e-01	3.37
1.0	6.7965e-02	53.66	6.7965e-02	23.23	3.2950e-01	6.75	3.4289e-01	3.49

Table 4.8: Errors and CPU cost of mixed FEM and TGM for ψ and ϕ at $t = 0.3$ with different δ.

δ	$\|I_{2h}\psi_h - \psi\|_h$	CPUs (FEM)	$\|I_{2h}\Psi_h - \psi\|_h$	CPUs (TGM)	$\|\phi_h - \phi\|_h$	CPUs (FEM)	$\|\Phi_h - \phi\|_h$	CPUs (TGM)
10^{-4}	6.0376e-02	137.73	6.1407e-02	56.41	4.2711e-03	25.56	4.3471e-03	10.97
10^{-3}	6.0491e-02	140.19	6.1306e-02	60.50	1.1662e-02	27.75	1.3217e-02	11.45
10^{-2}	6.0394e-02	136.69	6.0700e-02	54.73	3.7962e-02	27.10	4.0948e-02	11.18
10^{-1}	5.9192e-02	136.49	5.9236e-02	61.79	1.2186e-01	26.47	1.2229e-01	11.06
1.0	5.8648e-02	136.84	5.8655e-02	55.51	3.7619e-01	25.13	3.7623e-01	11.89

Table 4.9: Errors and CPU cost of mixed FEM and TGM for ψ and ϕ at $t = 0.5$ with different δ.

δ	$\|I_{2h}\psi_h - \psi\|_h$	CPUs (FEM)	$\|I_{2h}\Psi_h - \psi\|_h$	CPUs (TGM)	$\|\phi_h - \phi\|_h$	CPUs (FEM)	$\|\Phi_h - \phi\|_h$	CPUs (TGM)
10^{-4}	4.6021e-02	230.50	4.6081e-02	98.15	1.0713e-03	48.20	1.0670e-03	20.00
10^{-3}	4.5974e-02	217.78	4.6029e-02	100.16	1.4904e-03	52.46	1.5464e-03	22.13
10^{-2}	4.5826e-02	229.63	4.6029e-02	94.52	2.8325e-03	53.24	2.9732e-03	22.35
10^{-1}	4.5597e-02	232.82	4.5600e-02	90.30	3.1026e-03	53.34	3.1433e-03	23.05
1.0	4.5558e-02	236.06	4.5558e-02	97.28	2.3557e-03	54.09	2.3600e-03	24.47

Table 4.10: Errors and CPU cost of mixed FEM and TGM for ψ and ϕ at $t = 0.7$ with different δ.

δ	$\|I_{2h}\psi_h - \psi\|_h$	CPUs (FEM)	$\|I_{2h}\Psi_h - \psi\|_h$	CPUs (TGM)	$\|\phi_h - \phi\|_h$	CPUs (FEM)	$\|\Phi_h - \phi\|_h$	CPUs (TGM)
10^{-4}	4.0985e-02	295.55	4.1094e-02	133.04	1.2078e-03	73.34	1.2163e-03	28.22
10^{-3}	4.0921e-02	297.17	4.1025e-02	135.85	2.9043e-03	72.26	2.9395e-03	30.44
10^{-2}	4.0567e-02	290.01	4.0635e-02	133.75	8.3514e-03	67.23	8.4132e-03	30.90
10^{-1}	3.9679e-02	288.39	3.9694e-02	132.35	2.1835e-02	65.65	2.1863e-02	32.03
1.0	3.9312e-02	291.94	3.9316e-02	132.21	6.2504e-02	68.65	6.2511e-02	33.25

Table 4.11: Errors and CPU cost of mixed FEM and TGM for ψ and ϕ at $t = 1.0$ with different δ.

δ	$\|I_{2h}\psi_h - \psi\|_h$	CPUs (FEM)	$\|I_{2h}\Psi_h - \psi\|_h$	CPUs (TGM)	$\|\phi_h - \phi\|_h$	CPUs (FEM)	$\|\Phi_h - \phi\|_h$	CPUs (TGM)
10^{-4}	2.7923e-02	433.61	2.7936e-02	207.35	6.4724e-04	105.30	6.4644e-04	35.72
10^{-3}	2.7893e-02	425.83	2.7905e-02	214.07	9.2535e-04	103.45	9.2919e-04	36.37
10^{-2}	2.7798e-02	451.50	2.7805e-02	197.20	1.7552e-03	100.78	1.7714e-03	37.76
10^{-1}	2.7656e-02	492.35	2.7657e-02	191.78	1.8715e-03	105.07	1.8771e-03	40.22
1.0	2.7632e-02	475.67	2.7633e-02	199.25	1.4257e-03	100.90	1.4257e-03	39.57

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