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GENERALIZED JACOBI SPECTRAL GALERKIN METHOD FOR FRACTIONAL-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNELS*

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Abstract

For fractional Volterra integro-differential equations (FVIDEs) with weakly singular kernels, this paper proposes a generalized Jacobi spectral Galerkin method. The basis functions for the provided method are selected generalized Jacobi functions (GJFs), which can be utilized as natural basis functions of spectral methods for weakly singular FVIDEs when appropriately constructed. The developed method's spectral rate of convergence is determined using the L^∞-norm and the weighted L^2-norm. Numerical results indicate the usefulness of the proposed method.

Mathematics subject classification: 65L05, 65L20, 65L50.
Key words: Generalized Jacobi spectral Galerkin method, Fractional-order Volterra integro-differential equations, Weakly singular kernels, Convergence analysis.

1. Introduction

We consider in this paper the following fractional-order Volterra integro-differential equations with weakly singular kernels:

$$\begin{cases} {}_0D_t^\mu y(t) = y(t) + \displaystyle\int_0^t (t-\tau)^{-\nu} K(t,\tau) y(\tau) d\tau + g(t), \quad t \in I := [0,T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $0 < \mu < 1$, $0 < \nu < 1$, $K \in C(D)$ with $D := \{(t,\tau) : 0 \leqslant \tau \leqslant t \leqslant T\}$ and $g(t)$ is a continuous function. ${}_0D_t^\mu$ denotes the left-sided Reimann–Liouville fractional derivative of order μ (see the definition in (2.2a)).

The fractional calculus has made significant progress in both theory and practice, and its appearance and development have somewhat compensated for the shortcomings of the integer-order classical calculus. Fluid flow in porous materials, anomalous diffusion transport, acoustic

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wave propagation in viscoelastic materials, dynamics in self-similar structures, signal processing, financial theory, electric conductance of biological systems have all been better described using fractional calculus in the last two decades (see, e.g., [9,16,19,21,23]). In mathematical modeling of many physical phenomena, such as heat conduction in materials with memory in [20], many fractional-order Volterra integro-differential equations are used. Conduction, convection, and radiation are all examples of such equations (see, e.g., [1,3,22] and the references therein).

The three main difficulties in solving FVIDEs with weakly singular kernels in (1.1) are:

(i) fractional derivatives and integral operators are non-local operators, resulting in full matrices;

(ii) their solutions are often singular, making polynomial based approximations inefficient;

(iii) the solutions are usually singular near $t = 0$.

Spectral methods have been frequently utilized for numerical approximations [2,11,27,28] because it can provide extremely precise numerical approximations with fewer degrees of freedom. Well-designed spectral methods, in particular, appear to be particularly appealing for dealing with the above-mentioned challenges connected with FVIDEs with weakly singular kernels. For FVIDEs, polynomial based spectral methods have been developed (cf. [12,25,32,33,38] and the references therein). These approaches, on the other hand, rely on polynomial basis functions, which are not ideal for FVIDEs with non-smooth solutions at $t = 0$. Jacobi poly-fractonomials, which are defined as eigenfunctions of a fractional Sturm-Liouville problem, were first presented by Zayernouri and Karniadakis [35] to approximate the singular solutions. Chen *et al.* [4] employed generalized Jacobi functions (GJFs), which contain Jacobi poly-fractonomials as special cases, to create efficient Petrov-Galerkin methods for fractional PDEs. Following that, other authors devised spectral methods for fractional PDEs utilizing nodal GJFs [13,15,34,36,37].

When $\mu = 1$, the Eq. (1.1) is the classical Volterra integro-differential equations (VIDEs) with weakly singular kernels. Recently, many kinds of spectral collocation methods are proposed for solving Volterra integro equations (VIEs) with smooth kernels (cf. [7,17,29–31] and the references therein). To solve VIEs with weakly singular kernels, many attempts have been made to overcome the difficulties caused by the singularities of the solutions. For weakly singular VIEs, Chen and Tang [5,6] established spectral collocation methods. In [26], nonlinear VIEs with weakly singular kernels are solved using a generalized Jacobi-Galerkin method. In [18], linear VIDEs have been solved by Petrov–Galerkin method. Huang *et al.* [14] studied the supergeometric convergence of spectral collocation methods for weakly singular Volterra/Fredholm integral equations, etc.

The organization of this paper is as follows. In the next section, we introduce some useful properties of fractional calculus. In Section 3, we present the generalized Jacobi spectral Galerkin method for FVIDEs with weakly singular kernels in (1.1). Some useful lemmas for the convergence analysis are provided in Section 4. The convergence of the generalized Jacobi spectral Galerkin method is given in Section 5. We present in Section 6 some illustrative numerical results. Some concluding remarks are given in the last section.

2. Preliminaries

2.1. Fractional derivatives

We start with some preliminary definitions of fractional derivatives (see, e.g., [9,23]). To fix the idea, we restrict our attentions to the interval $\Lambda := [-1, 1]$.

For $\rho \in \mathbb{R}^+$ the left-sided and right-sided Reimann–Liouville integrals are respectively defined as

$$ {}_{-1}I_x^{\rho} y(x) = \frac{1}{\Gamma(\rho)} \int_{-1}^{x} \frac{y(s)}{(x-s)^{1-\rho}} ds, \quad x \in \Lambda, \tag{2.1a} $$

$$ {}_xI_1^{\rho} y(x) = \frac{1}{\Gamma(\rho)} \int_{x}^{1} \frac{y(s)}{(s-x)^{1-\rho}} ds, \quad x \in \Lambda, \tag{2.1b} $$

where $\Gamma(\cdot)$ is the usual Gamma function.

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided and right-sided Reimann–Liouville fractional derivative of order ν are defined by

$$ {}_{-1}D_x^{\nu} y(x) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dx^m} \int_{-1}^{x} \frac{y(s)}{(x-s)^{\nu-m+1}} ds, \quad x \in \Lambda, \tag{2.2a} $$

$$ {}_xD_1^{\nu} y(x) = \frac{(-1)^m}{\Gamma(m-\nu)} \frac{d^m}{dx^m} \int_{x}^{1} \frac{y(s)}{(s-x)^{\nu-m+1}} ds, \quad x \in \Lambda. \tag{2.2b} $$

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided and right-sided Caputo fractional derivative of order ν are defined by

$$ {}_{-1}^{C}D_x^{\nu} y(x) = \frac{1}{\Gamma(m-\nu)} \int_{-1}^{x} \frac{y^{(m)}(s)}{(x-s)^{\nu-m+1}} ds, \quad x \in \Lambda, \tag{2.3a} $$

$$ {}_{x}^{C}D_1^{\nu} y(x) = \frac{(-1)^m}{\Gamma(m-\nu)} \int_{x}^{1} \frac{y^{(m)}(s)}{(s-x)^{\nu-m+1}} ds, \quad x \in \Lambda. \tag{2.3b} $$

According to [9, Theorem 2.14], we have that for any absolutely integrable function f, and real $\nu \geqslant 0$,

$$ {}_{-1}D_x^{\nu}\,{}_{-1}I_x^{\nu} y(x) = y(x), \quad {}_xD_1^{\nu}\,{}_xI_1^{\nu} y(x) = y(x), \quad x \in \Lambda. \tag{2.4} $$

The following lemma shows the relationship between the Riemann-Liouville and Caputo fractional derivatives.

Lemma 2.1 ([9, 23]). *For $s \in [k-1, k)$ with $k \in \mathbb{N}$, we have*

$$ {}_{-1}D_x^{s} v(x) = {}_{-1}^{C}D_x^{s} v(x) + \sum_{j=0}^{k-1} \frac{v^{(j)}(-1)}{\Gamma(1+j-s)} (1+x)^{j-s}, \tag{2.5a} $$

$$ {}_xD_1^{s} v(x) = {}_{x}^{C}D_1^{s} v(x) + \sum_{j=0}^{k-1} \frac{(-1)^j v^{(j)}(1)}{\Gamma(1+j-s)} (1-x)^{j-s}. \tag{2.5b} $$

Remark 2.1. We observe immediately from (2.5) that for $s \in [k-1, k)$ with $k \in \mathbb{N}$,

$$ {}_{-1}D_x^{s} v(x) = {}_{-1}^{C}D_x^{s} v(x), \quad \text{if} \quad v^{(j)}(-1) = 0, \quad 0 \le j \le k-1, \tag{2.6a} $$

$$ {}_xD_1^{s} v(x) = {}_{x}^{C}D_1^{s} v(x), \quad \text{if} \quad v^{(j)}(+1) = 0, \quad 0 \le j \le k-1. \tag{2.6b} $$

2.2. Non-homogeneous initial conditions

For the case of non-homogeneous initial conditions when $y(0) = y_0 \neq 0$, we use the lifting a known solution method, in which we deconstruct the solution $y(t)$ into two halves as follows:

$$ y(t) = y_{\mathcal{H}}(t) + y_{\mathcal{D}}, \tag{2.7} $$

in which $y_{\mathcal{H}}(t)$ corresponds to the homogeneous solution and $y_{\mathcal{D}} = y_0$ is the nonzero initial condition, given in (1.1). We substitute (2.7) into (1.1) to get

$$\begin{cases} {}_0D_t^{\mu} y_{\mathcal{H}}(t) = y_{\mathcal{H}}(t) + \int_0^t (t-\tau)^{-\nu} K(t,\tau) y_{\mathcal{H}}(\tau) d\tau + \overline{g(t)}, \quad t \in (0,T], \\ y_{\mathcal{H}}(0) = 0, \end{cases} \tag{2.8}$$

where

$$\overline{g(t)} = g(t) + y_{\mathcal{D}} \left(\int_0^t (t-\tau)^{\nu} K(t,s) ds + 1 - \frac{1}{\Gamma(1-\mu) t^{\mu}} \right). \tag{2.9}$$

We can solve Eq. (2.8) of homogeneous initial value conditions for the weakly singular FVIDE equations with non-homogeneous initial value conditions. As a result, we only need to think about the case where $y(0) = 0$.

3. Generalized Jacobi Spectral Galerkin Method

3.1. Standard Jacobi polynomials

For $\alpha, \beta > -1$, let $J_n^{(\alpha,\beta)}(x)$, $x \in \Lambda := (-1,1)$ be the standard Jacobi polynomial of degree n, and denote the weight function $\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$. The set of Jacobi polynomials is a complete $L^2_{\omega^{(\alpha,\beta)}}(\Lambda)$-orthogonal system, i.e.,

$$\int_{-1}^1 J_n^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x) \omega^{(\alpha,\beta)}(x) dx = \gamma_m^{(\alpha,\beta)} \delta_{mn}, \tag{3.1}$$

in which δ_{mn} denotes the Kronecker function, and

$$\gamma_m^{(\alpha,\beta)} = \begin{cases} \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & m = 0, \\ \dfrac{2^{\alpha+\beta+1}}{(2m+\alpha+\beta+1)} \dfrac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\,\Gamma(m+\alpha+\beta+1)}, & m \geq 1. \end{cases} \tag{3.2}$$

In particular, $J_0^{(\alpha,\beta)}(x) = 1$.

For any integer $N \geq 0$, we denote by $\{x_j, \omega_j^{(\alpha,\beta)}\}_{j=0}^N$ the nodes and the corresponding Christoffel numbers of the standard Jacobi-Gauss interpolation on the interval Λ. Let $\mathcal{P}_N(\Lambda)$ be the set of polynomials of degree at most N on the interval Λ. Due to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\psi \in \mathcal{P}_{2N+1}$,

$$\int_{-1}^1 \psi(x) \omega^{(\alpha,\beta)}(x) dx = \sum_{j=0}^N \psi(x_j) \omega_j^{(\alpha,\beta)}. \tag{3.3}$$

3.2. Generalized Jacobi functions

The generalized Jacobi functions of degree n is defined by (cf. [4])

$$^{+}J_n^{(-\alpha,\beta)}(x) := (1-x)^{\alpha} J_n^{(\alpha,\beta)}(x) \quad \text{for} \quad \alpha > -1, \quad \beta \in \mathbb{R}, \tag{3.4}$$

$$^{-}J_n^{(\alpha,-\beta)}(x) := (1+x)^{\beta} J_n^{(\alpha,\beta)}(x) \quad \text{for} \quad \alpha \in \mathbb{R}, \quad \beta > -1 \tag{3.5}$$

for all $x \in \Lambda$ and $n \geq 0$. The orthogonal properties of the generalized Jacobi functions are as follows:

$$\int_{-1}^{1} {}^{+}J_n^{(-\alpha,\beta)}(x){}^{+}J_m^{(-\alpha,\beta)}(x)\omega^{(-\alpha,\beta)}(x)dx$$
$$= \int_{-1}^{1} {}^{-}J_n^{(\alpha,-\beta)}(x){}^{-}J_m^{(\alpha,-\beta)}(x)\omega^{(\alpha,-\beta)}(x)dx = \gamma_m^{(\alpha,\beta)}\delta_{nm},$$

where $\gamma_m^{(\alpha,\beta)}$ is defined in (3.2).

On the other hand, we define the finite-dimensional fractional-polynomial space

$$^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda) = \left\{\phi = (1+x)^{\beta}\psi : \psi \in \mathcal{P}_N\right\} = \operatorname{span}\left\{{}^{-}J_n^{(\alpha,-\beta)}(x) : 0 \leq n \leq N\right\}. \tag{3.6}$$

Because of (3.3), it follows that for any $\phi = (1+x)^{2\beta}\psi$ with $\psi \in \mathcal{P}_{2N+1}$,

$$\int_{-1}^{1} \phi(x)\omega^{(\alpha,-\beta)}(x)dx = \int_{-1}^{1} \psi(x)\omega^{(\alpha,\beta)}(x)dx$$
$$= \sum_{j=0}^{N} \psi(x_j)\omega_j^{(\alpha,\beta)} = \sum_{j=0}^{N}(1+x_j)^{-2\beta}\phi(x_j)\omega_j^{(\alpha,\beta)}. \tag{3.7}$$

Next, let $(u,v)_{\omega^{(\alpha,-\beta)}}$ and $\|u\|_{\omega^{(\alpha,-\beta)}}$ be the inner product and the norm of the space $L^2_{\omega^{(\alpha,-\beta)}}(\Lambda)$, respectively. We also introduce the following discrete inner product and norm on the interval Λ:

$$\begin{aligned} \langle u,v\rangle_{\omega^{(\alpha,-\beta)}} &= \sum_{j=0}^{N}(1+x_j)^{-2\beta}u(x_j)v(x_j)\omega_j^{(\alpha,\beta)}, \\ \|u\|_{N,\omega^{(\alpha,-\beta)}} &= \langle u,u\rangle^{\frac{1}{2}}_{\omega^{(\alpha,-\beta)}}. \end{aligned} \tag{3.8}$$

According to (3.7), for any $\phi,\psi \in {}^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda)$,

$$(\phi,\psi)_{\omega^{(\alpha,-\beta)}} = \langle\phi,\psi\rangle_{\omega^{(\alpha,-\beta)}}, \quad \|\phi\|_{\omega^{(\alpha,-\beta)}} = \|\phi\|_{N,\omega^{(\alpha,-\beta)}}. \tag{3.9}$$

Lemma 3.1 ([4]). *Let $s \in \mathbb{R}^+, n \in \mathbb{N}_0$ and $x \in \Lambda$. For $\alpha > 0, \beta \in \mathbb{R}$,*

$${}_xD_1^{\alpha}\left\{{}^{+}J_n^{(-\alpha,\beta)}(x)\right\} = \frac{\Gamma(n+\alpha+1)}{n!}{}^{+}J_n^{(0,\alpha+\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!}J_n^{(0,\alpha+\beta)}(x), \tag{3.10a}$$

For $\alpha \in \mathbb{R}, \beta > 0$,

$${}_{-1}D_x^{\beta}\left\{{}^{-}J_n^{(\alpha,-\beta)}(x)\right\} = \frac{\Gamma(n+\beta+1)}{n!}{}^{-}J_n^{(\alpha+\beta,0)}(x) = \frac{\Gamma(n+\beta+1)}{n!}J_n^{(\alpha+\beta,0)}(x). \tag{3.10b}$$

3.3. Generalized Jacobi spectral Galerkin method for problem (1.1)

To be able to apply the properties of orthogonal polynomials, we use the following interval transformation:

$$t = \frac{T}{2}(1+x), \quad \tau = \frac{T}{2}(1+s),$$

and let

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad {}_{-1}D_x^{\mu}u(x) = \left(\frac{T}{2}\right)^{\mu}{}_0D_t^{\mu}y\left(\frac{T}{2}(1+x)\right),$$
$$G(x) = \left(\frac{T}{2}\right)^{\mu} g\left(\frac{T}{2}(1+x)\right), \quad k(x,s) = \left(\frac{T}{2}\right)^{\mu-\nu+1} K\left(\frac{T}{2}(1+x), \frac{T}{2}(1+s)\right).$$

The fractional-order Volterra integro-differential equations weakly singular kernels (1.1) can then be transformed into the following form:

$$\begin{cases} {}_{-1}D_x^{\mu}u(x) = \xi u(x) + \int_{-1}^{x}(x-s)^{-\nu}k(x,s)u(s)ds + G(x), \quad x \in \Lambda, \\ u(-1) = 0, \end{cases} \tag{3.11}$$

where $\xi = (T/2)^{\mu}$.

We first define a linear integral operator $\chi : C(\Lambda) \to C(\Lambda)$ by

$$(\chi\psi)(x) := \int_{-1}^{x}(x-s)^{-\nu}k(x,s)\psi(s)ds.$$

In order to reduce the singularity caused by the fractional derivative, and also to apply Lemma 3.1, we choose $\alpha = -\nu, \beta = \mu$. Immediately after we let $u = u(x)$, $D^{\mu}u = {}_{-1}D_x^{\mu}u(x)$. Obviously, we know from (3.6) that

$${}^{-}\mathcal{F}^{(-\nu,-\mu)}(\Lambda) = \operatorname{span}\{{}^{-}J_n^{(-\nu,-\mu)}(x) : 0 \le n \le N\}.$$

Then, the generalized Jacobi spectral Galerkin scheme for problem (3.11) is to seek $u_N \in {}^{-}\mathcal{F}^{(-\nu,-\mu)}(\Lambda)$ such that

$$(D^{\mu}u_N, v_N)_{\omega^{(-\nu,-\mu)}} = (\xi u_N + \chi u_N + G, v_N)_{\omega^{(-\nu,-\mu)}}, \quad \forall v_N \in {}^{-}\mathcal{F}^{(-\nu,-\mu)}(\Lambda). \tag{3.12}$$

For simplicity of expression, we define

$$\phi_n(x) := {}^{-}J_n^{(-\nu,-\mu)}(x). \tag{3.13}$$

The numerical implementation of (3.12) is now described. For that purpose, we set

$$u(x) \approx u_N(x) = \sum_{n=0}^{N} a_n\phi_n(x). \tag{3.14}$$

Substituting (3.14) into (3.12) and taking $v_N = \phi_m(x)$, we obtain that for $0 \le m \le N$,

$$\begin{aligned} &\sum_{n=0}^{N} a_n(\phi_m, D^{\mu}\phi_n)_{\omega^{(-\nu,-\mu)}} \\ =& \sum_{n=0}^{N} a_n\xi(\phi_m,\phi_n)_{\omega^{(-\nu,-\mu)}} + \sum_{n=0}^{N} a_n(\phi_m, \chi\phi_n)_{\omega^{(-\nu,-\mu)}} + (\phi_m, G)_{\omega^{(-\nu,-\mu)}}. \end{aligned} \tag{3.15}$$

Set

$$\begin{aligned} &\mathbf{a} = (a_0, \cdots, a_N)^T, \qquad \mathbf{D}_{m,n} = (\phi_m, D^{\mu}\phi_n)_{\omega^{(-\nu,-\mu)}}, \\ &\mathbf{P}_{m,n} = \xi(\phi_m,\phi_n)_{\omega^{(-\nu,-\mu)}} = \left(\frac{T}{2}\right)^{\mu}\gamma_{mn}^{(-\nu,-\mu)}\delta_{mn}, \\ &\mathbf{V}_{m,n} = (\phi_m, \chi\phi_n)_{\omega^{(-\nu,-\mu)}}, \quad \mathbf{F} = (f_0, \cdots, f_N)^T, \\ &f_m = (\phi_m, G)_{\omega^{(-\nu,-\mu)}}, \qquad \mathbf{A} = \mathbf{D} - \mathbf{P} - \mathbf{V}. \end{aligned}$$

Then, the system (3.15) becomes

$$\mathbf{A}\mathbf{a} = \mathbf{F}. \tag{3.16}$$

In actual computation, we use the quadrature formula (3.8) and Lemma 3.1 to approximate the terms $\mathbf{D}_{m,n}$ and f_m, namely,

$$\begin{aligned}\mathbf{D}_{m,n} &= \left(\phi_m, \frac{\Gamma(n+\mu+1)}{n!} J_n^{(\mu-\nu,0)}\right)_{\omega^{(-\nu,-\mu)}} \\ &= \frac{\Gamma(n+\mu+1)}{n!} \sum_{j=0}^{N} (1+x_j)^{-2\mu} \phi_m(x_j) J_n^{(\mu-\nu,0)}(x_j) \omega_j^{(-\nu,\mu)}, \end{aligned} \tag{3.17}$$

and

$$f_m \approx \langle \phi_m, G\rangle_{\omega^{(-\nu,-\mu)}} = \sum_{j=0}^{N} (1+x_j)^{-2\mu} \phi_m(x_j) G(x_j) \omega_j^{(-\nu,\mu)}. \tag{3.18}$$

Finally, let us calculate $\mathbf{V}_{m,n}$. For this purpose, set

$$s(x,\theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}, \quad \theta \in \Lambda. \tag{3.19}$$

It is clear that

$$\begin{aligned}\chi\phi_n(x) &= \int_{-1}^{x} (x-s)^{-\nu} k(x,s) \phi_n(s) ds \\ &= \left(\frac{1+x}{2}\right)^{1-\nu} \int_{-1}^{1} (1-\theta)^{-\nu} k\big(x, s(x,\theta)\big) \phi_n\big(s(x,\theta)\big) d\theta. \end{aligned} \tag{3.20}$$

In virtue of (3.8) and (3.20), we obtain the following result:

$$\mathbf{V}_{m,n} \approx \left(\frac{1}{2}\right)^{1-\nu} \sum_{i,j=0}^{N} (1+x_i)^{1-2\mu-\nu} \phi_m(x_i) k\big(x_i, s(x_i,x_j)\big) \phi_n\big(s(x_i,x_j)\big) \omega_i^{(-\nu,\mu)} \omega_j^{(-\nu,0)}, \tag{3.21}$$

where $\{x_i\}_{i=0}^N$ and $\{x_j\}_{j=0}^N$ are the $(N+1)$-degree Jacobi–Gauss points corresponding to the weights $\{\omega_i^{(-\nu,\mu)}\}_{i=0}^N$ and $\{\omega_j^{(-\nu,0)}\}_{j=0}^N$, respectively.

4. Some Useful Lemmas

We will present some basic lemmas in this section, which are necessary for the derivation of the main results in the following section. First we define a weighted space $L^2_{\omega^{(\alpha,\beta)}}(\Lambda)$ as

$$L^2_{\omega^{(\alpha,\beta)}}(\Lambda) = \{v : v \text{ is measureable and } \|v\|_{\omega^{(\alpha,\beta)}} < \infty\}$$

with the inner product and norm

$$(u,v)_{\omega^{(\alpha,\beta)}} = \int_\Lambda u(x)v(x)\omega^{(\alpha,\beta)}(x)dx, \quad \|v\|_{\omega^{(\alpha,\beta)}} = \left(\int_\Lambda \omega^{\alpha,\beta}(x) v^2(x) dx\right)^{\frac{1}{2}}.$$

Furthermore, we define

$$H^m_{\omega^{(\alpha,\beta)}}(\Lambda) = \big\{u : D^k u \in L^2_{\omega^{(\alpha,\beta)}}(\Lambda),\ 0 \le k \le m\big\},$$

equipped with the norm

$$\|u\|_{H^m_{\omega^{(\alpha,\beta)}}(\Lambda)} = \left(\sum_{k=0}^{m} \left\|\frac{d^k u}{dx^k}\right\|^2_{\omega^{(\alpha,\beta)}}\right)^{\frac{1}{2}},$$

where $D^k v = d^k v/dx^k$.

Consider the $L^2_{\omega^{(\alpha,-\beta)}}$-orthogonal projection upon ${}^-\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda)$, defined by

$$\left({}^-\pi_N^{(\alpha,-\beta)}u-u,v_N\right)_{\omega^{(\alpha,-\beta)}}=0,\quad \forall v_N\in {}^-\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda). \tag{4.1}$$

To characterize the regularity of u, we introduce two non-uniformly weighted spaces involving fractional derivatives

$$\begin{aligned}
{}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda)&:=\left\{u\in L^2_{\omega^{(\alpha,-\beta)}}(\Lambda):{}_{-1}D_x^{\beta+l}u\in L^2_{\omega^{(\alpha+\beta+l,l)}}(\Lambda)\ \text{ for }\ 0\le l\le m\right\},\quad m\in\mathbb{N}_0,\\
{}^-\bar{\mathcal{B}}^m_{\alpha,\beta}(\Lambda)&:=\left\{u\in L^2_{\omega^{(\alpha,-\beta)}}(\Lambda):{}_{-1}D_x^{2\beta+l}u\in L^2_{\omega^{(\alpha+\beta+l,l)}}(\Lambda)\ \text{ for }\ 0\le l\le m\right\},\quad m\in\mathbb{N}_0.
\end{aligned}$$

We define space

$$\mathcal{C}^m_{\alpha,\beta}(\Lambda):=H^m_{\omega^{(\alpha,-\beta)}}(\Lambda)\cap{}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda)\cap{}^-\bar{\mathcal{B}}^m_{\alpha,\beta}(\Lambda).$$

Lemma 4.1 ([4]). *Let $\alpha>-1,\beta>0$, for any $u\in{}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda)$ with integer $0\le m\le N$, we have*

$$\left\|{}^-\pi_N^{(\alpha,-\beta)}u-u\right\|_{\omega^{(\alpha,-\beta)}}\le cN^{-(\beta+m)}\left\|D_-^{\beta+m}u\right\|_{\omega^{(\alpha+\beta+m,m)}}.$$

Lemma 4.2 ([2, 28]). *Suppose that $u\in H^m_{\omega^{(\alpha,\beta)}}(\Lambda)$ and $m\ge1$,*

$$\|u-\pi_Nu\|_\infty\le CN^{\frac{3}{4}-m}|u|_{H^{m;N}_{\omega^{(\alpha,\beta)}}(\Lambda)},$$

where $|u|_{H^{m,N}_{\omega^{(\alpha,\beta)}}(\Lambda)}$ denotes the seminorm defined by

$$|u|_{H^{m;N}_{\omega^{(\alpha,\beta)}}(\Lambda)}=\left(\sum_{k=\min(m,N+1)}^{m}\left\|\frac{d^ku}{dx^k}\right\|^2_{\omega^{(\alpha,\beta)}}\right)^{\frac12},$$

note that whenever $N\ge m-1$, one has

$$|u|_{H^{m;N}_{\omega^{(\alpha,\beta)}}(\Lambda)}=\|u^{(m)}\|_{L^2_{\omega^{(\alpha,\beta)}}(\Lambda)}=|u|_{H^m_{\omega^{(\alpha,\beta)}}(\Lambda)}.$$

Lemma 4.3 ([10]). *Suppose that $u\in L^2_{\omega^{(\alpha,\beta)}}(\Lambda)$, then*

$$\|\pi_Nu\|_{\omega^{(\alpha,\beta)}}\le C\|u\|_{\omega^{(\alpha,\beta)}},\quad \|\pi_Nu\|_\infty\le C\|u\|_\infty.$$

Next, we will introduce the Hölder space. Set $m\ge0$ and $\varrho\in(0,1), C^{m,\varrho}(\Lambda)$ consists of the m-times continuously differentiable function u and whose m-th derivatives are Hölder continuous with exponent ϱ. The norm is defined as follows:

$$\|u\|_{m,\varrho}=\sum_{k=0}^m\max_{z\in\Lambda}\left|\partial_z^ku(z)\right|+\sup_{\substack{z_1,z_2\in\Lambda\\ z_1\ne z_2}}\frac{|\partial_z^mu(z_1)-\partial_z^mu(z_2)|}{|z_1-z_2|^\varrho},$$

and if $\varrho=0$, then $C^{m,0}(\Lambda)$ represents the space of the m-times continuously derivative functions on Λ, it is also generally indicated by $C^m(\Lambda)$, and with norm $\|\cdot\|_m$.

Lemma 4.4 ([24]). *For a nonnegative integer r and $\kappa\in(0,1)$, there exists a constant $C_{r,\kappa}>0$ such that for any function $v\in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\mathcal{T}_Nv\in\mathcal{P}_N$ such that*

$$\|v-\mathcal{T}_Nv\|_\infty\le C_{r,\kappa}N^{-(r+\kappa)}\|v\|_{r,\kappa},$$

where $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}([-1,1])$ into $\mathcal{P}_N$, as stated in [24].

Lemma 4.5 ([8]). *Let $\kappa \in (0,1)$ and let $\mathcal{M}$ be defined by*

$$\big({}^{(\mu)}\mathcal{M}v\big)(x) = \int_{-1}^{x} (x-\tau)^{-\mu} K(x,\tau) v(\tau) d\tau.$$

Then, for any function $v \in C([-1,1])$, there exists a positive constant C such that

$$\frac{\big|{}^{(\mu)}\mathcal{M}v(x') - {}^{(\mu)}\mathcal{M}v(x'')\big|}{|x'-x''|} \le C \max_{x\in[-1,1]} |v(x)|,$$

under the assumption that $0 < \kappa < 1-\mu$, for any $x', x'' \in [-1,1]$ and $x' \neq x''$. This implies that

$$\|{}^{(\mu)}\mathcal{M}v\|_{0,\kappa} \le C \max_{x\in[-1,1]} |v(x)|, \quad 0 < \kappa < 1-\mu.$$

Lemma 4.6 (Grönwall Inequality). *Suppose $L \ge 0, 0 < \mu < 1$, and u and v are a non-negative, locally integrable functions defined on $[-1,1]$ satisfying*

$$u(x) \le v(x) + L \int_{-1}^{x} (x-\tau)^{-\mu} u(\tau) d\tau.$$

Then, there exists a constant $C = C(\mu)$ such that

$$u(x) \le v(x) + CL \int_{-1}^{x} (x-\tau)^{-\mu} v(\tau) d\tau, \quad -1 \le x \le 1.$$

If a nonnegative integrable function $E(x)$ satisfies

$$E(x) \le L \int_{-1}^{x} E(s) ds + J(x), \quad -1 < x \le 1,$$

where $J(x)$ is an integrable function, then

$$\begin{aligned} &\|E\|_{L^\infty(\Lambda)} \le C\|J\|_{L^\infty(\Lambda)}, \\ &\|E\|_{L^p_{\omega^{(\alpha,\beta)}}(\Lambda)} \le C\|J\|_{L^p_{\omega^{(\alpha,\beta)}}(\Lambda)}, \quad p \ge 1. \end{aligned}$$

Here and below, C denotes a positive constant which is independent of N.

5. Convergence Analysis

To make this section of the convergence analysis go more smoothly, we will first introduce the relational expression about $u(x)$. Since $u(-1) = 0$, we have

$${}_{-1}I_x^{\mu}\big({}_{-1}D_x^{\mu} u(x)\big) = u(x). \tag{5.1}$$

Then $u(x)$ can be expressed as

$$u(x) = \frac{1}{\Gamma(\mu)} \int_{-1}^{x} (x-s)^{\mu-1} {}_{-1}D_x^{\mu} u(s) ds. \tag{5.2}$$

Similarly, we define a linear integral operator $\lambda : C(\Lambda) \to C(\Lambda)$ by

$$(\lambda\psi)(x) := \frac{1}{\Gamma(\mu)} \int_{-1}^{x} (x-s)^{\mu-1} \psi(s) ds.$$

Hence, the problem (3.11) reads: Find $u = u(x)$ and $D^\mu u = D^\mu u(x)$ such that

$$\begin{aligned} &D^\mu u(x) = \xi u(x) + (\chi u)(x) + G(x), \\ &u(x) = (\lambda D^\mu u)(x). \end{aligned} \tag{5.3}$$

At this point, the problem (3.12) can be rewritten as: Find $u_N \in {}^{-}\mathcal{F}^{(-\nu,-\mu)}(\Lambda)$ such that

$$\begin{aligned} &(D^\mu u_N, v_N)_{\omega^{(-\nu,-\mu)}} = (\xi u_N + \chi u_N + G, v_N)_{\omega^{(-\nu,-\mu)}}, \\ &(u_N, v_N)_{\omega^{(-\nu,-\mu)}} = (\lambda D^\mu u_N, v_N)_{\omega^{(-\nu,-\mu)}}, \quad \forall v_N \in {}^{-}\mathcal{F}^{(-\nu,-\mu)}(\Lambda). \end{aligned} \tag{5.4}$$

Let $u_N^\mu = D^\mu u_N$ and $\pi_N = \pi_N^{(-\nu,-\mu)}$, according to (5.4) and the definition of the projection operator π_N, we obtain

$$\begin{aligned} &u_N^\mu = \xi u_N + \pi_N \chi u_N + \pi_N G, \\ &u_N = \pi_N \lambda u_N^\mu. \end{aligned} \tag{5.5}$$

Theorem 5.1. *Suppose that u_N is the generalized Jacobi spectral Galerkin solution determined by (5.4), if the solution u of (3.11) satisfies $u \in \mathcal{C}^m_{-\nu,\mu}(\Lambda)$, then we have the maximum error estimates*

$$\begin{aligned} &\|u - u_N\|_\infty \le CN^{\frac{3}{4}-m}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + \big|{}_{-1}D_x^\mu u\big|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big), \\ &\big\|{}_{-1}D_x^\mu u - u_N^\mu\big\|_\infty \le CN^{\frac{3}{4}-m}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + \big|{}_{-1}D_x^\mu u\big|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big). \end{aligned} \tag{5.6}$$

Proof. Let $e = u - u_N$, $e^\mu = D^\mu u - u_N^\mu$, the combination of (5.3) and (5.5) leads to

$$\begin{aligned} &D^\mu u - u_N^\mu = \xi(u - u_N) + \chi u - \pi_N \chi u_N + G - \pi_N G, \\ &u - u_N = \lambda D^\mu u - \lambda u_N^\mu. \end{aligned} \tag{5.7}$$

We can first get the following results by direct calculation:

$$\begin{aligned} \chi u - \pi_N \chi u_N &= \chi u - \pi_N \chi u + \pi_N \chi(u - u_N) \\ &= \chi u - \pi_N \chi u + \chi(u - u_N) + \big[\pi_N \chi(u - u_N) - \chi(u - u_N)\big] \\ &= (D^\mu u - \xi u - G) - \pi_N(D^\mu u - \xi u - G) + \chi(u - u_N) \\ &\quad + \big[\pi_N \chi(u - u_N) - \chi(u - u_N)\big] \\ &= (D^\mu u - \xi u - G) - \pi_N(D^\mu u - \xi u - G) + \chi e + (\pi_N \chi e - \chi e). \end{aligned} \tag{5.8}$$

We can, on the other hand, obtain

$$\begin{aligned} \lambda D^\mu u - \pi_N \lambda u_N^\mu &= \lambda D^\mu u - \pi_N \lambda D^\mu u + \pi_N \lambda\big(D^\mu u - u_N^\mu\big) \\ &= \lambda D^\mu u - \pi_N \lambda D^\mu u + \lambda\big(D^\mu u - u_N^\mu\big) \\ &\quad + \big[\pi_N \lambda(D^\mu u - u_N^\mu) - \lambda(D^\mu u - u_N^\mu)\big] \\ &= u - \pi_N u + \lambda e^\mu + \big(\pi_N \lambda e^\mu - \lambda e^\mu\big). \end{aligned} \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7), we have

$$\begin{aligned} e^\mu(x) &= \xi e(x) + \int_{-1}^x (x-s)^{-\nu} k(x,s) e(s) ds + D^\mu u - \pi_N D^\mu u \\ &\quad + \xi(\pi_N u - u) + (\pi_N \chi e - \chi e) \\ &= \xi e(x) + \int_{-1}^x k(x,s) e(s) ds + I_1 - \xi I_2 + I_3, \\ e(x) &= \frac{1}{\Gamma(\mu)} \int_{-1}^x (x-s)^{\mu-1} e^\mu(s) ds + I_2 + I_4, \end{aligned} \tag{5.10}$$

where

$$I_1 = D^\mu u - \pi_N D^\mu u, \quad I_2 = u - \pi_N u, \quad I_3 = \pi_N \chi e - \chi e, \quad I_4 = \pi_N \lambda e^\mu - \lambda e^\mu.$$

To establish a direct relationship between $e^\mu(x)$ and $e(x)$, we apply Dirichlet's formula, which states

$$\int_{-1}^{x}\int_{-1}^{\tau} \Phi(\tau, s) ds\, d\tau = \int_{-1}^{x}\int_{s}^{x} \Phi(\tau, s) d\tau\, ds. \tag{5.11}$$

Then, from (5.10) we can get

$$\begin{aligned} e^\mu(x) &= \xi e(x) + \int_{-1}^{x}\left(\int_{\tau}^{x} \frac{1}{\Gamma(\mu)} k(x,s) ds\right)(x-\tau)^{\mu-\nu-1} e^\mu(\tau) d\tau \\ &\quad + \int_{-1}^{x} (x-s)^{-\nu} k(x,s)\big(I_2(s) + I_4(s)\big) ds + I_1(x) - \xi I_2(x) + I_3(x) \\ &\le \xi |e(x)| + C\int_{-1}^{x} (x-\tau)^{\mu-\nu-1} |e^\mu|(\tau) d\tau + |I_1(x)| + C|I_2(x)| \\ &\quad + |I_3(x)| + C|I_4(x)|. \end{aligned} \tag{5.12}$$

In terms of Grönwall inequality and (5.12), we obtain

$$\|e^\mu(x)\|_\infty \le C\left(\|e(x)\|_\infty + \sum_{i=1}^{4} \|I_i\|_\infty\right). \tag{5.13}$$

On the other hand, combining Lemmas 4.4, 4.5 and (5.10), we have

$$\begin{aligned} \|e(x)\|_\infty &\le C\big\|{}^{(1-\mu)}\mathcal{M} e^\mu\big\|_\infty + \|I_2\|_\infty + \|I_4\|_\infty \\ &= C\big\|\big({}^{(1-\mu)}\mathcal{M} e^\mu - \mathcal{T}_N {}^{(1-\mu)}\mathcal{M} e^\mu\big)\big\|_\infty + \|I_2\|_\infty + \|I_4\|_\infty \\ &\le CN^{-\iota}\big\|{}^{(1-\mu)}\mathcal{M} e^\mu\big\|_{0,\iota} + \|I_2\|_\infty + \|I_4\|_\infty \\ &\le CN^{-\iota}\|e^\mu\|_\infty + \|I_2\|_\infty + \|I_4\|_\infty, \quad \iota \in (0,\mu). \end{aligned} \tag{5.14}$$

According to (5.13)-(5.14), we can get

$$\begin{aligned} \|e^\mu(x)\|_\infty &\le C\sum_{i=1}^{4} \|I_i\|_\infty, \\ \|e(x)\|_\infty &\le C\sum_{i=1}^{4} \|I_i\|_\infty. \end{aligned} \tag{5.15}$$

According to Lemma 4.2, we can get error estimates for I_1 and I_2 as

$$\begin{aligned} \|I_1\|_\infty &\le CN^{\frac{3}{4}-m} |D^\mu u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}, \\ \|I_2\|_\infty &\le CN^{\frac{3}{4}-m} |u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}. \end{aligned} \tag{5.16}$$

On the other hand, by Lemmas 4.3-4.5,

$$\begin{aligned} \|I_3\|_\infty &= \big\|(\pi_N - I)^{(\nu)}\mathcal{M} e\big\|_\infty = \big\|(\pi_N - I)\big({}^{(\nu)}\mathcal{M} e - \mathcal{T}_N {}^{(\nu)}\mathcal{M} e\big)\big\|_\infty \\ &\le \big\|\pi_N\big({}^{(\nu)}\mathcal{M} e - \mathcal{T}_N {}^{(\nu)}\mathcal{M} e\big)\big\|_\infty + \big\|{}^{(\nu)}\mathcal{M} e - \mathcal{T}_N {}^{(\nu)}\mathcal{M} e\big\|_\infty \\ &\le C\big\|{}^{(\nu)}\mathcal{M} e - \mathcal{T}_N {}^{(\nu)}\mathcal{M} e\big\|_\infty \le CN^{-\kappa}\big\|{}^{(\nu)}\mathcal{M} e\big\|_{0,\kappa} \\ &\le CN^{-\kappa}\|e\|_\infty, \quad \kappa \in (0, 1-\nu). \end{aligned} \tag{5.17}$$

Similarly, by combining Lemmas 4.3-4.5, we get

$$\begin{aligned}\|I_4\|_\infty &= \big\|(\pi_N - I)^{(1-\mu)}\mathcal{M}e^\mu\big\|_\infty \\ &= \big\|(\pi_N - I)(^{(1-\mu)}\mathcal{M}e^\mu - \mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^\mu)\big\|_\infty \\ &\le C\big\|^{(1-\mu)}\mathcal{M}e^\mu - \mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^\mu\big\|_\infty \\ &\le CN^{-\bar\kappa}\big\|^{(1-\mu)}\mathcal{M}e^\mu\big\|_{0,\bar\kappa} \\ &\le CN^{-\bar\kappa}\|e^\mu\|_\infty, \quad \bar\kappa \in (0,\mu). \end{aligned} \tag{5.18}$$

Together with (5.15)-(5.18), when N is large enough, we obtain

$$\begin{aligned} \|u-u_N\|_\infty &\le CN^{\frac{3}{4}-m}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + |D^\mu u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big), \\ \big\|D^\mu u - u_N^\mu\big\|_\infty &\le CN^{\frac{3}{4}-m}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + |D^\mu u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big). \end{aligned} \tag{5.19}$$

Therefore, Theorem 5.1 is proved. □

Following that, we study the weighted L^2 error estimate, which is based on the L^∞ error estimate.

Theorem 5.2. *Suppose that u_N is the generalized Jacobi spectral Galerkin solution determined by* (5.4)*, if the solution u of* (3.11) *satisfies $u \in \mathcal{C}^m_{-\nu,\mu}(\Lambda)$, then we have the $\|\cdot\|_{\omega^{(-\nu,-\mu)}}$ error estimates*

$$\begin{aligned} \|u-u_N\|_{\omega^{(-\nu,-\mu)}} &\le cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}} + \big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}}\Big) \\ &\quad + CN^{\frac{3}{4}-m-\gamma}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + \big|{}_{-1}D_x^\mu u\big|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big), \\ \big\|{}_{-1}D_x^\mu u - u_N^\mu\big\|_{\omega^{(-\nu,-\mu)}} &\le cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}} + \big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}}\Big) \\ &\quad + CN^{\frac{3}{4}-m-\gamma}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + \big|{}_{-1}D_x^\mu u\big|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big), \end{aligned}$$

where $\gamma = \min\{\eta, \bar\eta\}$, the value ranges of η and $\bar\eta$ are $\eta \in (0, 1-\nu)$ and $\bar\eta \in (0,\mu)$, respectively.

Proof. Now we investigate the $\|\cdot\|_{\omega^{(-\nu,-\mu)}}$-error estimates. It follows from (5.12) and Grönwall inequality that

$$\|e^\mu(x)\|_{\omega^{(-\nu,-\mu)}} \le C\left(\|e(x)\|_{\omega^{(-\nu,-\mu)}} + \sum_{i=1}^4 \|I_i\|_{\omega^{(-\nu,-\mu)}}\right). \tag{5.20}$$

Similarly, combining Lemmas 4.4, 4.5 and (5.10), we have

$$\begin{aligned} \|e(x)\|_{\omega^{(-\nu,-\mu)}} &\le C\big\|^{(1-\mu)}\mathcal{M}e^\mu\big\|_\omega + \|I_2\|_\omega + \|I_4\|_\omega \\ &= C\big\|\big(^{(1-\mu)}\mathcal{M}e^\mu - \mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^\mu\big)\big\|_\omega + \|I_2\|_\omega + \|I_4\|_\omega \\ &\le C\big\|\big(^{(1-\mu)}\mathcal{M}e^\mu - \mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^\mu\big)\big\|_\infty + \|I_2\|_\omega + \|I_4\|_\omega \\ &\le CN^{-\sigma}\big\|^{(1-\mu)}\mathcal{M}e^\mu\big\|_{0,\sigma} + \|I_2\|_\omega + \|I_4\|_\omega \\ &\le CN^{-\sigma}\|e^\mu\|_\omega + \|I_2\|_\omega + \|I_4\|_\omega, \quad \sigma \in (0,\mu). \end{aligned} \tag{5.21}$$

From (5.20) and (5.21), we can get

$$\begin{aligned}
&\|e^{\mu}(x)\|_{\omega^{(-\nu,-\mu)}} \le C\sum_{i=1}^{4}\|I_i\|_{\omega^{(-\nu,-\mu)}},\\
&\|e(x)\|_{\omega^{(-\nu,-\mu)}} \le C\sum_{i=1}^{4}\|I_i\|_{\omega^{(-\nu,-\mu)}}.
\end{aligned} \tag{5.22}$$

Due to Lemma 4.1, we have

$$\begin{aligned}
&\|I_1\|_{\omega^{(-\nu,-\mu)}} \le cN^{-(\mu+m)}\big\|{}_{-1}D_x^{\mu+m}\big({}_{-1}D_x^{\mu}u\big)\big\|_{\omega^{(\mu-\nu,m)}},\\
&\|I_2\|_{\omega^{(-\nu,-\mu)}} \le cN^{-(\mu+m)}\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}}.
\end{aligned} \tag{5.23}$$

Because $u(-1)=0$, ${}_{-1}D_x^{\mu+m}({}_{-1}D_x^{\mu}u) = {}_{-1}D_x^{2\mu+m}u$ in the above formula holds. Therefore,

$$\|I_1\|_{\omega^{(-\nu,-\mu)}} \le cN^{-(\mu+m)}\big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}}. \tag{5.24}$$

It follows from Lemmas 4.4-4.5, we can get the $\|\cdot\|_{\omega^{(-\nu,-\mu)}}$-error estimate for I_3.

$$\begin{aligned}
\|I_3\|_{\omega^{(-\nu,-\mu)}} &= \big\|(\pi_N - I)^{(\nu)}\mathcal{M}e\big\|_{\omega} = \big\|(\pi_N - I)\big({}^{(\nu)}\mathcal{M}e - \mathcal{T}_N{}^{(\nu)}\mathcal{M}e\big)\big\|_{\omega}\\
&\le \big\|\pi_N\big({}^{(\nu)}\mathcal{M}e - \mathcal{T}_N{}^{(\nu)}\mathcal{M}e\big)\big\|_{\omega} + \big\|{}^{(\nu)}\mathcal{M}e - \mathcal{T}_N{}^{(\nu)}\mathcal{M}e\big\|_{\omega}\\
&\le C\big\|{}^{(\nu)}\mathcal{M}e - \mathcal{T}_N{}^{(\nu)}\mathcal{M}e\big\|_{\infty} \le CN^{-\eta}\big\|{}^{(\nu)}\mathcal{M}e\big\|_{0,\eta}\\
&\le CN^{-\eta}\|e\|_{\infty}, \quad \eta \in (0, 1-\nu).
\end{aligned} \tag{5.25}$$

We find from Lemmas 4.3-4.5 that

$$\begin{aligned}
\|I_4\|_{\omega^{(-\nu,-\mu)}} &= \big\|(\pi_N - I)^{(1-\mu)}\mathcal{M}e^{\mu}\big\|_{\omega}\\
&= \big\|(\pi_N - I)\big({}^{(1-\mu)}\mathcal{M}e^{\mu} - \mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^{\mu}\big)\big\|_{\omega}\\
&\le C\big\|{}^{(1-\mu)}\mathcal{M}e^{\mu} - \mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^{\mu}\big\|_{\infty}\\
&\le CN^{-\bar{\eta}}\big\|{}^{(1-\mu)}\mathcal{M}e^{\mu}\big\|_{0,\bar{\eta}}\\
&\le CN^{-\bar{\eta}}\|e^{\mu}\|_{\infty}, \quad \bar{\eta} \in (0,\mu).
\end{aligned} \tag{5.26}$$

Let $\gamma = \min\{\eta,\bar{\eta}\}$, the combination of Theorem 5.1, (5.22)-(5.26) obtains

$$\begin{aligned}
\|u - u_N\|_{\omega^{(-\nu,-\mu)}} \le{}& cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}} + \big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}}\Big)\\
&+ CN^{\frac{3}{4}-m-\gamma}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + \big|{}_{-1}D_x^{\mu}u\big|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big),\\
\big\|{}_{-1}D_x^{\mu}u - u_N^{\mu}\big\|_{\omega^{(-\nu,-\mu)}} \le{}& cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}} + \big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(\mu-\nu+m,m)}}\Big)\\
&+ CN^{\frac{3}{4}-m-\gamma}\Big(|u|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)} + \big|{}_{-1}D_x^{\mu}u\big|_{H^{m;N}_{\omega^{(-\nu,-\mu)}}(\Lambda)}\Big),
\end{aligned}$$

provided N is large enough. Hence, the Theorem 5.2 is proved. □

6. Numerical Experiments

We provide some numerical examples to back up our analysis. To measure the efficiency of the results, $\|\cdot\|_{\infty}$ and $\|\cdot\|_{\omega}$ errors are used to assess the efficiency of the method. Matlab is used to perform all of the calculations.

Example 6.1. We consider the following the fractional-order Volterra integro-differential equations with weakly singular kernels:

$$\begin{cases} {}_0D_t^{\mu}y(t) = y(t) + \int_0^t (t-\tau)^{-\nu} y(\tau)d\tau + g(t), & \mu \in (0,1), \\ y(0) = 0 \end{cases} \tag{6.1}$$

with

$$g(t) = \frac{\Gamma(n+\delta+1)}{\Gamma(n+\delta+1-\mu)} t^{n+\delta-\mu} - t^{n+\delta} - t^{n+1+\delta-\nu} B(n+1+\delta, 1-\nu),$$

where $0 \le \delta \le 1$, and $B(\cdot,\cdot)$ is the Beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

This problem has an unique solution $y(t) = t^{n+\delta}$. The weighted function ω is chosen as $\omega = (1-x)^{-\nu}(1+x)^{-\mu}$ with $\nu = 0.4$. The solution interval is $t \in [0,1]$, and the exact solution is $y(t) = t^{5.6}$, indicating that $n = 5$ and $\delta = 0.6$. In Fig. 6.1 (left), we plot the discrete L^2_ω errors and the maximum errors of (6.1) when $\mu = 0.5$. It is shown that the numerical errors decay exponentially as N increases. Fig. 6.1 (right) illustrates the numerical result of the generalized Jacobi spectral Galerkin method approximation solution for $N = 20$ and exact solution of (6.1) when $\mu = 0.5$, which are found in excellent agreement.

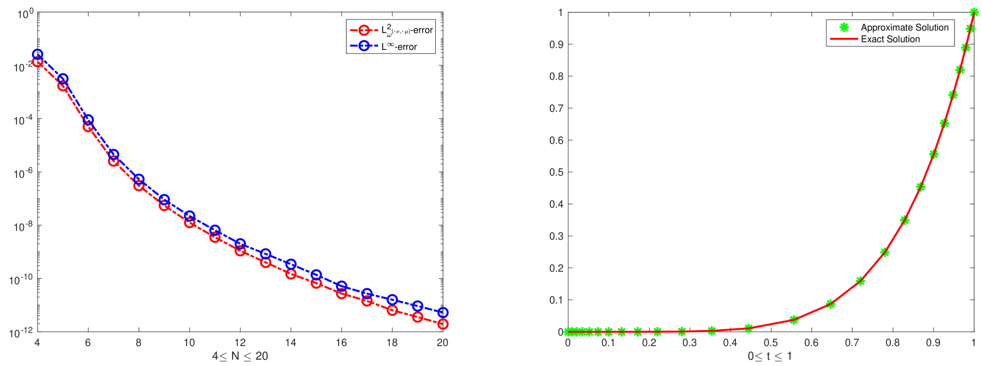

Fig. 6.1. Left: The L^∞ error and L^2_ω error versus N. Right: Numerical and exact solution of $y(t) = t^{5+0.6}$ with $n = 5$ and $\delta = 0.6$.

Table 6.1 lists the L^2_ω error of Example 6.1 when $\mu = 0.1, 0.5, 0.9$. It is shown that the generalized Jacobi spectral Galerkin method has achieved higher accuracy when $N = 4$, and as N increases, L^2_ω error of the numerical solution decreases rapidly. The numerical results demonstrate the high accuracy and effectiveness of the generalized Jacobi spectral Galerkin method.

Example 6.2. Consider next the linear FVIDEs with weakly singular kernel

$$\begin{cases} {}_0D_t^{\frac{1}{2}}y(t) = y(t) + \int_0^t (t-\tau)^{-\frac{1}{3}} y(\tau)d\tau + g(t), & t \in [0,1], \\ y(0) = 0. \end{cases} \tag{6.2}$$

Table 6.1: Example 6.1. $\|\cdot\|_\omega$ errors, for $0<\mu<1$.

N	$\mu=0.1$	$\mu=0.5$	$\mu=0.9$
4	2.7473×10^{-2}	1.3714×10^{-2}	8.0803×10^{-3}
6	1.8039×10^{-4}	5.0158×10^{-5}	1.0420×10^{-5}
8	6.6915×10^{-7}	3.0373×10^{-7}	8.9949×10^{-8}
10	2.4505×10^{-8}	1.2542×10^{-8}	4.1183×10^{-9}
12	2.0394×10^{-9}	1.0970×10^{-9}	3.8020×10^{-10}
14	2.7471×10^{-10}	1.4852×10^{-10}	5.2791×10^{-11}
16	5.1257×10^{-11}	2.7135×10^{-11}	9.6690×10^{-12}
18	1.2071×10^{-12}	6.4287×10^{-12}	2.3170×10^{-13}
20	3.3602×10^{-12}	1.9441×10^{-12}	7.2167×10^{-13}

We choose $g(t)$ such that the solution y of (6.2) is given by $y(t)=t^3\cos(t)$. We implement the numerical scheme (3.15) based on the generalized Jacobi spectral Galerkin method to solve this example. In Fig. 6.2 (left), numerical errors of (6.2) are plotted for $4\le N\le 20$ in both L^∞ and L^2_ω-norms. The numerical results of the generalized Jacobi spectral Galerkin method approximation solution for $N=20$ and the exact solution of (6.2) are shown in Fig. 6.2 (right), and they are found to be in excellent agreement. We again see that the observed convergence rate agrees with the expected rate.

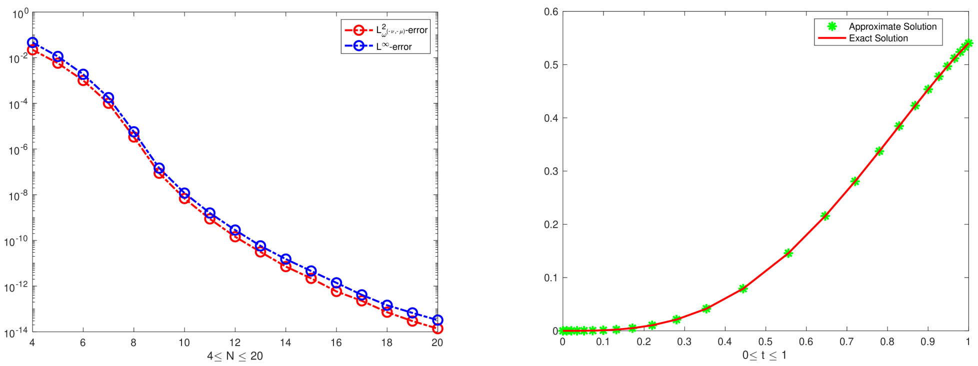

Fig. 6.2. Left: The L^∞ error and L^2_ω error versus N. Right: Numerical and exact solution of $y(t)=t^3\cos(t)$.

7. Conclusion

A generalized Jacobi spectral Galerkin method for fractional-order Volterra integro-differential equations with weakly singular kernels is proposed in this paper. The GJFs are chosen to match the leading singularity of the underlying problem, resulting in higher performance than the polynomial basis. The error estimates for the generalized Jacobi spectral Galerkin are established. Despite the solution singularity, numerical experiments demonstrated that the proposed method can provide very accurate results.

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