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Growth of Solutions of Higher Order Complex Linear Differential Equations in an Angular Domain of Unit Disc

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Abstract. We study the growth of solutions of higher order complex differential equations in an angular domain of the unit disc instead of the whole unit disc. Some conditions on coefficient functions, which will guarantee all non-trivial solutions of the higher order differential equations have fast growing, are found in this paper.

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1 Introduction and main results

For a function f meromorphic in the unit disc $\Delta=\{z:|z|<1\}$, the order of growth is given by

$$\rho(f)=\limsup_{r\to 1^-}\frac{\log^+ T(r,f)}{\log\frac{1}{1-r}}.$$

If f is an analytic function in Δ, then the order of growth of f is often given by

$$\rho_M(f)=\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f)}{\log\frac{1}{1-r}},$$

where $M(r,f)=\max\limits_{\substack{|z|=r\\ z\in\Delta}}|f(z)|$. It follows from [17, Theorem V.13] that

$$\rho(f)\le\rho_M(f)\le\rho(f)+1.$$

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Observe that there exists f such that $\rho(f)\neq\rho_M(f)$; for example, $f(z)=\exp\{(\frac{1}{1-z})^\lambda\}$ satisfies $\rho(f)=\lambda-1$ and $\rho_M(f)=\lambda$, where $\lambda>1$ is a constant, which can be found in [17, p. 205].

For the function of fast growth in Δ, we also need the definition of iterated $n-$order, which can be found in [3]. It is defined by

$$\rho_n(f)=\limsup_{r\to1^-}\frac{\log_n^+ T(r,f)}{\log\frac{1}{1-r}},$$

where $n\geq 1$ is integer, $\log_1^+ x=\log^+ x=\max\{\log x,0\}$, $\log_{n+1}^+ x=\log^+\log_n^+ x$. If f is an analytic function in Δ, then the iterated n-order is also given by

$$\rho_{M,n}(f)=\limsup_{r\to1^-}\frac{\log_{n+1}^+ M(r,f)}{\log\frac{1}{1-r}}.$$

Obviously,

$$\rho_1(f)\leq\rho_{M,1}(f)\leq\rho_1(f)+1$$

for any analytic functions in Δ. However, it follows from Proposition 2.2.2 in [14] that $\rho_n(f)=\rho_{M,n}(f)$ for $n\geq 2$. In general, $\rho_2(f)$ or $\rho_{M,2}(f)$ are called hyper-order of f in Δ. In this paper, we assume that the reader is familiar with the fundamental results and standard notation of the Nevanlinna's theory of meromorphic functions in Δ, see [13] and [21] for more details.

The meromorphic functions in Δ can be divided into the following three cases:

(1) Bounded type: $T(r,f)=O(1)$ as $r\to1^-$;

(2) Rational type: $T(r,f)=O(\log\frac{1}{1-r})$ as $r\to1^-$ and f does not belong to (1);

(3) Admissible in Δ:

$$\limsup_{r\to1^-}\frac{T(r,f)}{\log\frac{1}{1-r}}=\infty.$$

It is always interested in studying the growth of solutions of linear differential equations in the unit disc by using the Nevanlinna's theory of meromorphic functions. The analysis of slowly growing solutions has been studied in [5,8,9,11,12,15]. Fast growth of solutions are considered in [1,3,4,8,10]. There are a few results in studying the growth of solutions of differential equations in an angular domain. One of our main purpose of this paper is to investigate the properties in an angular domain of solutions of linear differential equation of the form

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.1}$$

where $A_0(z)\not\equiv 0,A_1(z),\ldots,A_k(z)$ are analytic functions in Δ.

In [8], Heittokangas studied the growth of solutions of second order linear differential equations and obtained the following result.