Journal of Computational Mathematics
Vol.32, No.5, 2014, 507–521.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1405-m4347

DIRECTIONAL DO-NOTHING CONDITION FOR THE NAVIER-STOKES EQUATIONS*

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Abstract

The numerical solution of flow problems usually requires bounded domains although the physical problem may take place in an unbounded or substantially larger domain. In this case, artificial boundaries are necessary. A well established artificial boundary condition for the Navier-Stokes equations discretized by finite elements is the "do-nothing" condition. The reason for this is the fact that this condition appears automatically in the variational formulation after partial integration of the viscous term and the pressure gradient. This condition is one of the most established outflow conditions for Navier-Stokes but there are very few analytical insight into this boundary condition. We address the question of existence and stability of weak solutions for the Navier-Stokes equations with a "directional do-nothing" condition. In contrast to the usual "do-nothing" condition this boundary condition has enhanced stability properties. In the case of pure outflow, the condition is equivalent to the original one, whereas in the case of inflow a dissipative effect appears. We show existence of weak solutions and illustrate the effect of this boundary condition by computation of steady and non-steady flows.

Mathematics subject classification: 35M12, 65M60, 76D03, 76D05.
Key words: Boundary conditions, Navier-Stokes, Outflow condition, Existence.

1. Introduction

The numerical solution of the Navier-Stokes equations usually requires bounded domains even though the physical problem may take place in an unbounded or substantially larger domain. Therefore, it is necessary to establish artificial boundaries. In fluid dynamics, such artificial boundaries often have an "outflow character". The most established outflow boundary condition for finite element discretization of the Navier-Stokes equations is the so-called "do-nothing" condition. That is because this condition appears automatically due to partial integration of the viscous term and the pressure gradient. If no further boundary integral is added to the variational formulation, the "do-nothing" condition is automatically built in. This formulation was already used by Glowinski [9] and by Gresho [8]. Since then, this condition became the most established outflow condition for Navier-Stokes.

* Received January 2, 2013 / Revised version received April 14, 2014 / Accepted May 26, 2014 / Published online August 22, 2014 /

However, there is hardly any analytical insight into the do-nothing boundary condition. The mere question of whether steady weak solutions for Navier-Stokes equations exist together with this do-nothing condition remains unsolved even in 2D. One of the few articles on this topic addresses different variational formulations and its relation to strong formulations with corresponding outflow conditions [10]. Again, the existence of weak solutions for large data remains a vital question for this boundary condition.

In this work, we address the existential question regarding solutions for the Navier-Stokes equations in combination with a "directional do-nothing" (DDN) condition which leads, on one hand, to enhanced stability. On the other hand, however, when looking at the pure outflow, this particular boundary condition is equivalent to a classical do-nothing condition. The difference to the classical do-nothing condition (CDN) is a nonlinear correction, leading to enhanced stability. This implies that there is no need for any smallness condition on the data to ensure existence. We prove in particular the existence of weak solutions and stability in the H^1-norm plus additional boundary control. This additionally implies uniqueness of small solutions. We analyze the steady case and draw a sketch of proof for the evolutionary system.

In [4], Bruneau and Fabrie presented an entire class of alternative outflow conditions involving several parameters. The DDN condition analyzed in this paper, is obtained for a particular choice of those parameters. A few years later, some investigation has been done by Neustupa and Feistauer in [15, 16]. Although this condition has fundamental advantages compared to the CDN condition, this boundary condition is not very popular yet. Our paper is aimed at stimulating the application of this interesting relation by concisely proving its existence, as well as presenting some new analytical and numerical arguments as to why the nonlinear correction can be useful for Navier-Stokes equations.

As usual for Navier-Stokes, uniqueness of obtained steady solutions for large data can not be addressed here. The theory [7,19] delivers some results about un-uniqueness and uniqueness properties. Hence, the answer concerning the uniqueness in our case is highly nontrivial. It may depend on the particular forcing and the geometry of the domain.

In the computational part of this work, we illustrate the quality of the "directional do-nothing" condition for steady and for non-steady flows and compare it with the CDN condition. As a result, we will see that the modified outflow condition reproduces the solution of the standard outflow condition in the case of pure outflow, and it exhibits too much inflow. This is, of course, a favorable property for an outflow condition.

The outline of the paper is as follows: In Section 2 we present the two types of outflow conditions CDN and DDN, and show why it is impossible to answer the question of existence regarding solutions for the CDN condition. In Section 3 we show stability and existence of solutions for the DDN condition. The time-dependent case is treated in Section 4. Finally, we show in Section 5 the effect of the DDN by means of numerical examples and illustrate the differences to the CDN condition.

2. Outflow Conditions for Navier-Stokes

We consider the stationary incompressible Navier-Stokes equation in the domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$,

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \operatorname{div} \mathbb{T}(\boldsymbol{v}, p) = \boldsymbol{f} \quad \text{in } \Omega\,, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega\,. \tag{2.2}$$