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THE GENERALIZED LOCAL HERMITIAN AND SKEW-HERMITIAN SPLITTING ITERATION METHODS FOR THE NON-HERMITIAN GENERALIZED SADDLE POINT PROBLEMS*

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Abstract

For large and sparse saddle point problems, Zhu studied a class of generalized local Hermitian and skew-Hermitian splitting iteration methods for non-Hermitian saddle point problem [M.-Z. Zhu, Appl. Math. Comput. 218 (2012) 8816–8824]. In this paper, we further investigate the generalized local Hermitian and skew-Hermitian splitting (GLHSS) iteration methods for solving non-Hermitian generalized saddle point problems. With different choices of the parameter matrices, we derive conditions for guaranteeing the convergence of these iterative methods. Numerical experiments are presented to illustrate the effectiveness of our GLHSS iteration methods as well as the preconditioners.

Mathematics subject classification: 65F10, 65F50.
Key words: Generalized saddle point problems, Hermitian and skew-Hermitian matrix splitting, Iteration method, Convergence.

1. Introduction

Consider the following two-by-two block linear systems of the form

$$\begin{bmatrix} A & B^* \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times m}$, $x,\ f \in \mathbb{C}^n$, $y,\ g \in \mathbb{C}^m$ with $m \le n$. B^* denotes the conjugate transpose of the matrix B.

The two-by-two block linear system (1.1) is often called as a generalized saddle point problem with $C \neq O$ and a saddle point problem with $C = O$, which is important and arises in a large number of scientific and engineering applications, such as the field of computational fluid dynamics [17], mixed finite element approximations of elliptic partial differential equations [10], restrictively preconditioned conjugate gradient methods [26, 28], interior point methods in constrained optimization [7]. For more applications or a comprehensive survey one can refer to [9].

As is known, there are two kinds of methods to solve the linear systems: direct methods and iterative methods. Direct methods are widely employed when the size of the coefficient matrix is not too large, and are usually regarded as the robust methods. However, frequently, matrices A and B are large and sparse, so iterative methods, such as Uzawa type methods

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[3, 6, 11, 12, 13, 15, 16, 19, 20, 22, 23, 31], HSS-type iteration methods [1, 2, 4, 5, 27], as well as preconditioned Krylov subspace iteration methods [14, 18], become more attractive than direct methods for solving the systems (1.1). In the case of A being a Hermitian positive definite, a large amount of efficient methods as well as their numerical properties have been studied in the literature. Bai et al. [2] established the Hermitian and skew-Hermitian splitting (HSS) iteration method and a class of preconditioned Hermitian and skew-Hermitian splitting (PHSS) iteration method [5] for general non-Hermitian linear system, which naturally result in the HSS-type iteration methods for solving the non-Hermitian(generalized) saddle point problems, i.e., the case of A being non-Hermitian. For example, Benzi and Golub [8] directly applied the HSS iteration technique to the non-Hermitian (generalized) saddle point problems. A preconditioned Hermitian and skew-Hermitian (PHSS) method for solving saddle point problem (1.1) was further presented by Bai et al. [5] and the accelerated Hermitian and skew-Hermitian (AHSS) splitting methods by Bai and Golub [1]. The more HSS-type iteration methods can be found in [21, 24, 25, 30]. Recently, Jiang et al. [21] proposed a local Hermitian and skew-Hermitian splitting (LHSS) iteration method and a modified LHSS iteration method (MLHSS) for the non-Hermitian saddle point problems with matrix $C = O$ by adding the proper parameter matrices to the Hermitian and skew-Hermitian parts of A and (2,2) zero block matrix in the split matrices, respectively. Furthermore, Zhu [32] investigated a generalized local Hermitian and skew-Hermitian splitting method (GLHSS) by adding one more parameter matrix to the (2,1) position in the first matrix of the splitting. These iteration methods can be applied to solve the non-Hermitian generalized saddle point problems with the restriction condition that C is Hermitian positive semi-definite by transforming the case of $C \neq O$ into its equivalent form of $C = O$. These mean that these methods are not appropriate for solving the non-Hermitian generalized saddle point problems with C being Hermitian positive definite. To bridge this gap, in this paper we use the same parameter matrices strategy as in [32] and propose a generalized local Hermitian and non-Hermitian splitting iteration method (still called GLHSS method) for the non-Hermitian generalized saddle point problems with the matrix C being Hermitian positive definite. The convergence properties are also discussed.

The rest of the paper is organized as follows. Section 2 gives the GLHSS iteration method for non-Hermitian generalized saddle point problems and the convergence properties are discussed under certain conditions for the case C being a special Hermitian positive definite. In Sections 3, we derive several algorithms by different choices of the parameter matrices. We simply describe the effectiveness of the GLHSS splitting presented in this paper as a preconditioner for preconditioned Krylov subspace methods such as GMRES method in Section 4. Section 5 provides some numerical experiments to illustrate our theory and some concluding remarks are given in Section 6.

2. The GLHSS Iteration Methods

Let $A = H + S$ be the Hermitian and skew-Hermitian splitting of A in which $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ are the Hermitian and skew-Hermitian parts of A, respectively. A is called an non-Hermitian positive definite matrix if $H = \frac{1}{2}(A + A^*)$ is positive definite, or simply, A is positive definite. In particular, A is called being Hermitian dominant if $\|H\| > \|S\|$. From now on, unless otherwise stated, we always assume that the matrix A is positive and Hermitian dominant.

For the coefficient matrix in the two-by-two linear system (1.1), we make the following