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HIGH ORDER LOCAL DISCONTINUOUS GALERKIN METHODS FOR THE ALLEN-CAHN EQUATION: ANALYSIS AND SIMULATION*

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Abstract

In this paper, we present a local discontinuous Galerkin (LDG) method for the Allen-Cahn equation. We prove the energy stability, analyze the optimal convergence rate of $k+1$ in L^2 norm and present the $(2k+1)$-th order negative-norm estimate of the semi-discrete LDG method for the Allen-Cahn equation with smooth solution. To relax the severe time step restriction of explicit time marching methods, we construct a first order semi-implicit scheme based on the convex splitting principle of the discrete Allen-Cahn energy and prove the corresponding unconditional energy stability. To achieve high order temporal accuracy, we employ the semi-implicit spectral deferred correction (SDC) method. Combining with the unconditionally stable convex splitting scheme, the SDC method can be high order accurate and stable in our numerical tests. To enhance the efficiency of the proposed methods, the multigrid solver is adapted to solve the resulting nonlinear algebraic systems. Numerical studies are presented to confirm that we can achieve optimal accuracy of $\mathcal{O}(h^{k+1})$ in L^2 norm and improve the LDG solution from $\mathcal{O}(h^{k+1})$ to $\mathcal{O}(h^{2k+1})$ with the accuracy enhancement post-processing technique.

Mathematics subject classification: 65M60, 35K55, 35L02
Key words: Local discontinuous Galerkin method, Allen-Cahn equation, Energy stability, Convex splitting, Spectral deferred correction, A priori error estimate, Negative norm error estimate.

1. Introduction

In this paper, we develop a local discontinuous Galerkin (LDG) method and consider error estimates of the LDG method for the Allen-Cahn equation

$$u_t - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \tag{1.1}$$

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with the initial condition

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \tag{1.2}$$

in a bounded domain with dimension $d \leq 3$. We assume that periodic boundary conditions are given. It is well-known that the Allen-Cahn equation is a gradient flow with the Liapunov energy functional

$$\mathscr{J}_\varepsilon(u) = \int_\Omega \Phi_\varepsilon(u) d\boldsymbol{x}, \quad \Phi_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}F(u), \tag{1.3}$$

where $F(u)$ is always positive and $f(u) = F'(u)$. A typical form of $F(u)$ is

$$F(u) = \frac{1}{4}(u^2-1)^2, \quad f(u) = u^3 - u. \tag{1.4}$$

As in [25], we shall impose a constraint on the potential function $F(u)$ by requiring $f(u)$ to satisfy

$$\max_{u \text{ solves Allen-Cahn}} |f'(u)| \leq L, \tag{1.5}$$

where L is a positive constant.

The Allen-Cahn equation (1.1) was originally introduced by Allen and Cahn [1] to describe the motion of anti-phase boundaries in crystalline solids. The function u represents the concentration of one of the two metallic components of the alloy and the positive parameter ε is called the diffuse interface width parameter. Recently, it has been applied to a wide range of problems such as the motion by mean curvature flows [14] and crystal growth [26]. In particular, it has become a basic model equation for the diffuse interface approach developed to study phase transitions and interfacial dynamics in materials science [5].

Various numerical methods have been developed to solve the Allen-Cahn equation. We refer the readers to [6,7] for finite difference method. Feng *et al.* [15] developed an a posteriori error estimate for finite element approximations of the Allen-Cahn equation. Quasi-optimal a posteriori error estimates in $L^\infty(0,T;L^2(\Omega))$ was derived for finite element approximation in [2]. The numerical approximations of the celebrated Allen-Cahn equation and related diffuse interface models were studied in [34]. Yang [33] introduced a stabilized semi-implicit (in time) scheme and a splitting scheme for the equation. Feng *et al.* [13] recently presented the analysis for the fully discrete interior penalty discontinuous Galerkin (IP-DG) methods for the Allen-Cahn equation. In [16], the first- and second-order implicit-explicit schemes with parameters for solving the Allen-Cahn equation were investigated. Feng, Tang and Yang [17] combined the semi-implicit spectral deferred correction (SDC) method with energy stable convex splitting technique to solve a series of phase field models.

In this paper, we present an LDG method for the Allen-Cahn equation and prove its energy stability, where the energy is defined in (1.3). In addition, the optimal priori error estimate is also proved in L^2 norm for the LDG scheme. By employing a technical dual argument, we obtain an a priori error estimate in the negative-order norm for smooth solutions of Allen-Cahn equation, which is $2k+1$, higher than the $(k+1)$-th order in L^2-norm, where k $(k \geq 1)$ is the highest degree polynomial used in the approximation. This negative norm error estimate is very essential for the accuracy enhancement post-processing technique [19,20]. Additionally, we present numerical studies which confirm that we can achieve optimal accuracy of $\mathcal{O}(h^{k+1})$ in L^2 norm and improve the LDG solution from $\mathcal{O}(h^{k+1})$ to $\mathcal{O}(h^{2k+1})$ with the accuracy enhancement post-processing technique.