FINITE DIFFERENCE METHOD FOR A NONLINEAR WAVE EQUATION*

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1. Differential Equation and Difference Equation

For a nonlinear wave equation in viscous flow and elastic mechanics, the existence of its solution has been explored in [1—4]. In this paper, we consider the finite difference method for the initial-boundary value problem of this nonlinear wave equation. We establish prior estimates for the solution of the difference equations on the basis of the prior estimates we prove the convergence and stability of the difference solution and the existence and uniqueness of the classical solution of the differential equation.

We consider the following initial-boundary problem

$$\begin{cases} u_{tt}-u_{xx}=(\sigma(u_x))_x+u_{xxt}-f(u), \quad 0<x<1,\ 0<t\leqslant T, & (1.1)\\ u|_{t=0}=u_0(x),\ u_t|_{t=0}=u_1(x), & (1.2)\\ u(0,t)=u(1,t)=0, & (1.3)\end{cases}$$

where $\sigma(p)$ and $f(u)$ are given functions, and $u_0(x)$ and $u_1(x)$ are known functions. On the interval $[0, 1]$, the step size of space is h, and the points of the net are $x_0=0$, $x_1=h$, $\cdots$, $x_J=1$. The step size of time is k. We use the following symbols of difference and norm:

$$(u_j^n)_x=\frac{1}{h}(u_{j+1}^n-u_j^n), \qquad (u_j^n)_{\bar x}=\frac{1}{h}(u_j^n-u_{j-1}^n),$$

$$(u_j^n)_{\hat x}=\frac{1}{2h}(u_{j+1}^n-u_{j-1}^n), \qquad (u_j^n)_{\tilde x}=\frac{1}{h}(u_{j+\frac12}^n-u_{j-\frac12}^n),$$

$$(u_j^n)_{\bar t}=\frac{1}{k}(u_j^n-u_j^{n-1}), \qquad (u_j^n, v_j^n)=h\sum_{j=0}^{J-1}u_j^n\cdot v_j^n,$$

$$\|u_j^n\|^2=h\sum_{j=0}^{J-1}(u_j^n)^2, \qquad \|u_j^n\|_{L_\infty}=\sup_{0<j<J-1}|u_j^n|.$$

In this paper we use C_i and K_i to denote positive constants.

For the problem (1.1)—(1.3), we give the following implicit scheme

$$\begin{cases} (u_j^{n+1})_{t\bar t}-(u_j^{n+1})_{x\bar x}=(\sigma((u_j^{n+1})_{\hat x}))_{\tilde x}+(u_j^{n+1})_{x\bar x\bar t}-f(u_j^{n+1}), \\ \qquad\qquad j=1,2,\cdots,J-1, \quad n=0,1,\cdots, & (1.4)\\ u_j^0=u_0(x_j), \quad (u_j^0)_t=u_1(x_j), & (1.5)\\ u_0^n=u_J^n=0, & (1.6)\end{cases}$$

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where u_j^{-1} is solved by the initial condition (1.5). We add quantities u_j^{-2}, u_{-1}^n and u_{-2}^n to solution u_j^n, $1\leqslant j\leqslant J-1$, $n=0, 1, \cdots$. The quantities u_j^{-2}, u_{-1}^n and u_{-2}^n are defined by the following formulas respectively

$$(u_j^0)_{t\bar t}-(u_j^0)_{x\bar x}=(\sigma((u_j^0)_{\bar x}))_x+(u_j^0)_{x\bar x t}-f(u_j^0), \tag{1.7}$$

$$(u_0^{n+1})_{t\bar t}-(u_0^{n+1})_{x\bar x}=(\sigma((u_0^{n+1})_{\bar x}))_x+(u_0^{n+1})_{x\bar x t}-f(u_0^{n+1}), \tag{1.8}$$

$$(u_0^{n+1})_{t\bar t x}-(u_0^{n+1})_{x\bar x x}=(\sigma((u_0^{n+1})_{\bar x}))_{xx}+(u_0^{n+1})_{x\bar x x t}-[f(u_0^{n+1})]_x. \tag{1.9}$$

2. Basic Estimations and Existence of the Solution for the Differential Equation

Lemma 1. *Assume that* (i) $Q(p)=\int_0^p\sigma(r)dr\geqslant 0$, $\sigma(p)\in C^1$, $\sigma'(p)\geqslant 0$, $p\in(-\infty, \infty)$; (ii) $F(u)=\int_0^u f(r)dr\geqslant 0$, $f(u)\in C^1$, $f'(u)\geqslant 0$, $u\in(-\infty, \infty)$; (iii) $u_0(x)\in H_0^1$, $u_1(x)\in L_2$, $\int_0^1 Q(u_0'(x))dx<\infty$, $\int_0^1 F(u_0(x))dx<\infty$. *Then we have the estimations*

$$\|(u_j^n)_{\bar t}\|\leqslant C_1,\quad \|(u_j^n)_x\|\leqslant C_1,\quad \|u_j^n\|\leqslant C_1,$$

$$\|u_j^n\|_{L_\infty}\leqslant C_1,\quad k\sum_{\alpha=1}^n\|(u_j^\alpha)_{x\bar t}\|^2\leqslant C_2,$$

$$h\sum_{j=0}^{J-1}Q((u^n_{j+\frac12})_{\bar x})\leqslant C_2,\quad h\sum_{j=0}^{J-1}F(u_j^n)\leqslant C_2,\quad 0\leqslant nk\leqslant T.$$

Proof. Multiplying (1.4) by $(u_j^{n+1})_{\bar t}$ and taking the inner product we have

$$((u_j^{n+1})_{t\bar t}, (u_j^{n+1})_{\bar t})-((u_j^{n+1})_{x\bar x}, (u_j^{n+1})_{\bar t})$$
$$=((\sigma((u_j^{n+1})_{\bar x}))_x, (u_j^{n+1})_{\bar t})+((u_j^{n+1})_{x\bar x\bar t}, (u_j^{n+1})_{\bar t})-(f(u_j^{n+1}), (u_j^{n+1})_{\bar t}). \tag{2.1}$$

We deduce the terms of the above formula as follows

$$((u_j^{n+1})_{t\bar t}, (u_j^{n+1})_{\bar t})\geqslant\frac12(\|(u_j^{n+1})_{\bar t}\|^2)_{\bar t},\quad -((u_j^{n+1})_{x\bar x}, (u_j^{n+1})_{\bar t})\geqslant\frac12(\|(u_j^{n+1})_x\|^2)_{\bar t},$$

$$-((\sigma((u_j^{n+1})_{\bar x}))_x, (u_j^{n+1})_{\bar t})=(\sigma((u^{n+1}_{j+\frac12})_{\bar x}), (u^{n+1}_{j+\frac12})_{\bar x\bar t})\geqslant h\sum_{j=0}^{J-1}[Q((u^{n+1}_{j+\frac12})_{\bar x})]_{\bar t},$$

$$((u_j^{n+1})_{x\bar x\bar t}, (u_j^{n+1})_{\bar t})=-\|(u_j^{n+1})_{x\bar t}\|^2,$$

$$(f(u_j^{n+1}), (u_j^{n+1})_{\bar t})\geqslant h\sum_{j=0}^{J-1}[F(u_j^{n+1})]_{\bar t}.$$

Thus it follows from (2.1) that

$$\frac12(\|(u_j^{n+1})_{\bar t}\|^2)_{\bar t}+\frac12(\|(u_j^{n+1})_x\|^2)_{\bar t}+h\sum_{j=0}^{J-1}[Q((u^{n+1}_{j+\frac12})_{\bar x})]_{\bar t}$$
$$+\|(u_j^{n+1})_{x\bar t}\|^2+h\sum_{j=0}^{J-1}[F(u_j^{n+1})]_{\bar t}\leqslant 0,$$

$$\|(u_j^{n+1})_{\bar t}\|^2+\|(u_j^{n+1})_x\|^2+2h\sum_{j=0}^{J-1}Q((u^{n+1}_{j+\frac12})_{\bar x})+2k\sum_{\alpha=1}^{n+1}\|(u_j^\alpha)_{x\bar t}\|^2+2h\sum_{j=0}^{J-1}F(u_j^{n+1})$$
$$\leqslant\|(u_j^0)_{\bar t}\|^2+\|(u_j^0)_x\|^2+2h\sum_{j=0}^{J-1}Q((u^0_{j+\frac12})_{\bar x})+2h\sum_{j=0}^{J-1}F(u_j^0). \tag{2.2}$$

From (2.2) and the conditions of the lemma we obtain

$$\|(u_j^{n+1})_{\bar t}\|^2+\|(u_j^{n+1})_x\|^2\leqslant K_1.$$

By Sobolev's embedding theorem, the following inequalities holds

$$\|(u_j^n)_{\bar t}\|\leqslant C_1,\quad\|(u_j^n)_x\|\leqslant C_1,\quad\|u_j^n\|\leqslant C_1,\quad\|u_j^n\|_{L_\infty}\leqslant C_1.$$