ON THE RATE OF OVERCONVERGENCE OF THE GENERALIZED ENESTRÖM-KAKEYA FUNCTIONAL FOR POLYNOMIALS*

RICHARD S. VARGA[1)]

(Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242, USA)

WU WEN-DA (吴文达)

(Beijing Center for International Economic Information, Beijing, China)

Abstract

The classical Eneström-Kakeya Theorem, which provides an upper bound for the moduli of the zeros of any polynomial with positive coefficients, has been recently extended by Anderson, Saff and Varga to the case of any complex polynomial having no zeros on the ray $[0, +\infty)$. Their extension is sharp in the sense that, given such a complex polynomial $p_n(z)$ of degree $n\geqslant 1$, a sequence of multiplier polynomials $\{Q_{m_i}(z)\}_{i=1}^{\infty}$ can be found for which the Eneström-Kakeya upper bound, applied to the products $Q_{m_i}(z)\cdot p_n(z)$, converges, in the limit as i tends to ∞, to the maximum of the moduli of the zeros of $p_n(z)$. Here, the rate of convergence of these upper bounds (to the maximum of the moduli of the zeros of $p_n(z)$) is studied. It is shown that the obtained rate of convergence is best possible.

§ 1. Introduction

With π_n denoting the set of all complex polynomials of degree exactly n, and with

$$\pi_n^+ := \left\{ p_n(z) = \sum_{j=0}^{n} a_j z^j:\ a_j > 0 \text{ for all } j = 0, 1, \cdots, n \right\}, \tag{1.1}$$

a useful form of the classical Eneström-Kakeya Theorem[3,5], due in fact to Eneström[3], is the following

Theorem A. *For any* $p_n(z) = \sum_{j=0}^{n} a_j z^j$ *in* π_n^+ *with* $n\geqslant 1$, *define*

$$\alpha[p_n] := \min_{0\leqslant i<n}\left\{\frac{a_i}{a_{i+1}}\right\} \quad \text{and} \quad \beta[p_n] := \max_{0\leqslant i<n}\left\{\frac{a_i}{a_{i+1}}\right\}. \tag{1.2}$$

Then, all the zeros of $p_n(z)$ *lie in the annulus*

$$\alpha[p_n] \leqslant |z| \leqslant \beta[p_n]. \tag{1.3}$$

Evidently, if

$$\rho(p_n) := \max\{|z_j| : p_n(z_j) = 0\}$$

denotes the spectral radius of any complex polynomial $p_n(z)$ in π_n with $n\geqslant 1$, then the Eneström-Kakeya Theorem asserts that

$$\beta[p_n] \geqslant \rho(p_n), \text{ for all } p_n \in \pi_n^+, \text{ for all } n\geqslant 1. \tag{1.4}$$

* Received January 28, 1985.

1) Research supported in part by the Air Force Office of Scientific Research and the Department of Energy.

The sharpness of this inequality in (1.4) had already been studied in 1913 by Hurwitz[4]. For a recent (corrected) form of Hurwitz's original contribution which gives the precise conditions on $p_n(z)$ in π_n^+ so that equality holds in (1.4), see [1].

To go beyond the case of polynomials having only positive real coefficients, as treated in (1.4), consider more generally any complex polynomial $p_n(z)$ in π_n with $n\geqslant 1$, and suppose that a multiplier polynomial $Q_m(z)$ in π_m can be found such that $Q_m(z)\cdot p_n(z)$ is in π_{m+n}^+. Then, applying the Eneström–Kakeya Theorem to the product $Q_m(z)p_n(z)$ gives (cf. (1.4)) $\beta[Q_mp_n]\geqslant\rho(Q_mp_n)$, but as $\rho(Q_mp_n)\geqslant\rho(p_n)$, then

$$\beta[Q_mp_n]\geqslant\rho(p_n). \tag{1.5}$$

This upper bound $\beta[Q_mp_n]$ for $\rho(p_n)$ is called a generalized Eneström–Kakeya functional for $p_n(z)$, when such a multiplier polynomial exists. On setting

$$\hat{\pi}_n:=\{p_n(z)\in\pi_n:\ p_n(z)\text{ has no zeros on the ray }[0,+\infty)\}, \tag{1.6}$$

$$\text{for any } n\geqslant 1,$$

it was shown in Anderson, Saff and Varga [2, Prop. 1] that the existence of such a multiplier polynomial $Q_m(z)\in\pi_m$ for $p_n(z)$ for which $Q_m(z)\cdot p_n(z)\in\pi_{m+n}^+$, is equivalent with $p_n\in\hat{\pi}_n$. Moreover, for $p_n\in\hat{\pi}_n$, it easily follows (cf. [2]) that there exists a least nonnegative integer m_0 (depending on p_n) such that the set

$$\omega_m(p_n):=\{Q_m(z)\in\pi_m:\ Q_m(z)p_n(z)\in\pi_{m+n}^+\}$$

is nonempty for each $m\geqslant m_0$. Now,

$$\tau_m=\tau_m(p_n):=\inf\{\beta[Q_mp_n]:\ Q_m\in\omega_m(p_n)\},\quad m\geqslant m_0, \tag{1.7}$$

gives the optimal (least) upper bound estimate of $\rho(p_n)$ of this generalized Eneström–Kakeya functional, when restricted to polynomial multipliers $Q_m(z)$ of degree m. Moreover, with (1.5) and (1.7), it is evident that the $\tau_m(p_n)$'s are monotone decreasing:

$$\tau_{m_0}(p_n)\geqslant\tau_{m_0+1}(p_n)\geqslant\tau_{m_0+2}(p_n)\geqslant\cdots\geqslant\rho(p_n). \tag{1.8}$$

Because of the inequalities of (1.8), it is natural to ask if the sequence $\{\tau_m(p_n)\}_{m=m_0}^{\infty}$ tends to $\rho(p_n)$, as $m\to\infty$. An affirmative answer to this question, established in [2, Theorem 1], is stated as

Theorem B. *For any $p_n(z)$ in $\hat{\pi}_n$ with $n\geqslant 1$,*

$$\lim_{m\to\infty}\tau_m(p_n)=\rho(p_n). \tag{1.9}$$

Another question that can be asked is to characterize those elements $p_n(z)$ in $\hat{\pi}_n$ (with $n\geqslant 1$) for which there exists some positive integer

$$m_1=m_1(p_n)$$

such that

$$\tau_m(p_n)=\rho(p_n)$$

for all $m\geqslant m_1(p_n)$. To answer this question, it is convenient to define the subset $\hat{\hat{\pi}}_n$ of $\hat{\pi}_n$ (for each $n\geqslant 1$) by