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# NUMERICAL SOLUTION OF NON-STEADY STATE POROUS FLOW FREE BOUNDARY PROBLEMS*

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## Abstract

The aim of this paper is the study of the convergence of a finite element approximation for a variational inequality related to free boundary problems in non-steady fluid flow through porous media. There have been many results in the stationary case, for example, the steady dam problems ([3, 1]), the steady flow well problems[6], etc. In this paper we shall deal with the axisymmetric non-steady porous flow well problem. It is well known that by means of Torelli's transform this problem, similar to the non-steady rectangular dam problem, can be reduced to a variational inequality, and the existence, uniqueness and regularity of the solution can be obtained ([12, 7]). Now we study the numerical solution of this variational inequality.

The main results are as follows:

1. We establish new regularity properties for the solution $W$ of the variational inequality. We prove that $W\in L^\infty(0,\ T;\ H^2(D))$, $\gamma_0 W\in L^\infty(0,\ T;\ H^2(\Gamma_n))$ and $D_t\gamma_0 W\in L^2(0,\ T;\ H^1(\Gamma_n))$ (see Theorem 2.5). Friedman and Torelli[7] obtained $W\in L^2(0,\ T;\ H^2(D))$. Our new regularity properties will be used for error estimation.

2. We prove that the error estimate for the finite element solution of the variational inequality is

$$\{\sum_{i=1}^{N}\|W^i-W_h^i\|^2_{H^1(D)}\Delta t\}^{1/2}=O(h+\Delta t^{1/2})$$

(see Theorem 3.4). In the stationary case the error estimate is $\|W-W_h\|_{H^1(D)}=O(h)$ ([3, 6]).

3. We give a numerical example and compare the result with the corresponding result in the stationary case.

The results of this paper are valid for the non-steady rectangular dam problem with stationary or quasi-stationary initial data (see [7], p. 534).

## § 1. Introduction

In this section we state the non-steady porous flow well problem and the related results.

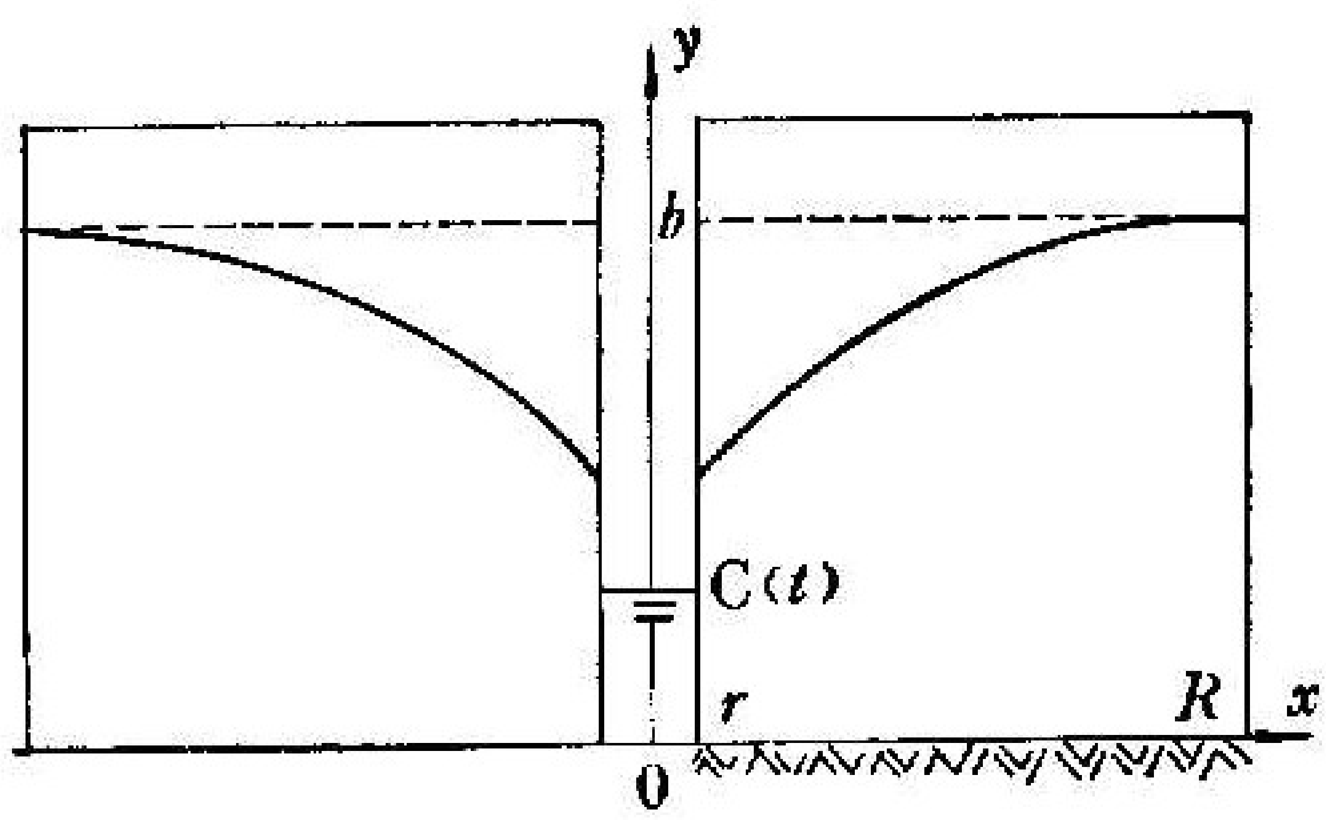


Fig. 1

1.1. Statement of the problem.

The non-steady state problem to be considered is shown in Figure 1. An axisymmetric well of radius $r$ is sunk into a soil aquifer of depth $b$ and radius $R$. The bottom of the aquifer is impervious. The outer boundary of the aquifer adjoins a catchment area and the hydraulic head $u(x,\ t)$ is equal to the constant $b_0$ along this boundary. $[0, T]$, with $T>0$, is the time interval during which the filtration process is studied. $C(t)$ is the water level in the well. We assume

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that $C(0)=b_0$ and $0<C(t)\leqslant b_0$, $\forall t\in[0, T]$. The water-air interface is a free boundary. $\varphi(x, t)$ represents the height of the free boundary. We suppose that $u(0, t)=b_0$, $\forall t\in[0, T]$. Finally we assume that the water is incompressible and the porou medium is homogeneous.

The mathematical problem can now be formulated as follows (see [2]):

**Problem 1.1.** We look for a triplet $\{\varphi, \Omega, u\}$ such that:

i) $\varphi$ is a *regular* function defined in $[r, R]\times[0, T]$, satisfying

$$\begin{cases} 0<\varphi(x, t)\leqslant b_0, & \forall(x, t)\in[r, R]\times[0, T], \\ \varphi(r, t)\geqslant C(t), \quad \varphi(R, t)=b_0, & \forall t\in[0, T], \\ \varphi(x, 0)=b_0, & \forall x\in[r, R]; \end{cases} \tag{1.1}$$

ii) $\Omega$ is defined by the relation:

$$\Omega=\{(x, y, t);\ r<x<R,\ 0<t<T,\ 0<y<\varphi(x, t)\}; \tag{1.2}$$

iii) $u$ is a *regular* function defined in $\bar{\Omega}$ such that:

$$Eu\equiv(xu_x)_x+xu_{yy}=0, \quad \text{in } \Omega, \tag{1.3}$$

$$\begin{cases} u(r, y, t)=C(t) & \text{if } 0\leqslant y\leqslant C(t),\ 0<t\leqslant T, \\ u(r, y, t)=y & \text{if } C(t)<y\leqslant\varphi(r, t),\ 0<t\leqslant T, \\ u(R, y, t)=b_0 & \text{if } 0\leqslant y\leqslant b_0,\ 0<t\leqslant T, \\ u_y(x, 0, t)=0 & \text{if } r<x<R,\ 0<t\leqslant T. \end{cases} \tag{1.4}$$

On the free boundary

$$\Sigma=\{(x, y, t);\ r<x<R,\ 0<t<T,\ y=\varphi(x, t)\},$$

$u$ satisfies the relations

$$\begin{cases} u(x, y, t)=y, \\ u_x^2+u_y^2-u_y=u_t. \end{cases} \tag{1.5}$$

This problem corresponds to the non-steady rectangular dam problem with stationary initial data. Furthermore we suppose that

$$C(t)\in C^1(0, T), \qquad C'(t)>-1. \tag{1.6}$$

1.2. Formulation as a variational inequality.

In this section we reformulate Problem 1.1 as a variational inequality.

Let

$$D=\{(x, y);\ r<x<R,\ 0<y<b\},$$

$$Q=D\times(0, T),$$

$$\Gamma_1=\{(x, y);\ x=r,\ 0<y<b\},$$

$$\Gamma_2=\{(x, y);\ x=R,\ 0<y<b\},$$

$$\Gamma_n=\{(x, y);\ r<x<R,\ y=0\}.$$

Set

$$\Gamma_d=\partial D\backslash\Gamma_n.$$

We introduce the functions