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SOME ADVANCES IN THE STUDY OF ERROR EXPANSION FOR FINITE ELEMENTS*

I. Eigenvalue Error Expansion

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Abstract

For the eigenvalue problem on a smooth domain we prove that the Ric hardson extrapolation increases the accuracy from second to third order for linear finite elements, and from fourth to fifth order for quadratic finite elements, without modification of the scheme near the boundary.

§ 1. Introduction

As an introductory model problem the simple eigenvalue problem

$$-\Delta u=\lambda u \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\quad \int_\Omega u^2\,dx=1 \tag{1}$$

on a smooth domain $\Omega\subset R^2$ will be investigated in the first part of this paper.

Let $T_h=\{K\}$ be a regular triangulation of Ω of width h with all its boundary vertices on $\partial\Omega$. Corresponding to T_h, we define the following finite element space of degree 1 or 2,

$$S_h=\{V_h\in C(\Omega^h)\,|\,V_h \text{ linear/quadratic on each } K\in T_h,\ V_h\equiv 0 \text{ on } \partial\Omega^h\},$$
$$\Omega^h=\cup\{K\in T_h\}.$$

The finite element eigenvalue problem associated with (1) is determined by

$$(\nabla u_h,\nabla\varphi_h)=\lambda_h(u_h,\varphi_h),\quad \forall\varphi_h\in S_h, \tag{2}$$
$$u_h\in S_h,\quad (u_h,u_h)=1.$$

We recall the eigenvalue errors:

$$\lambda_h-\lambda=\begin{cases}O(h^2) & \text{for linear elements,}\\ O(h^4) & \text{for quadratic elements}^{1)}.\end{cases}$$

Our purpose is to prove further error expansions

$$\lambda_h-\lambda=\begin{cases}h^2e+O(h^3) & \text{for linear elements,} & (3)\\ h^4e+O(h^5) & \text{for quadratic elements}^{1)}. & (4)\end{cases}$$

The second part of this paper develops to the solution problems

$$-\Delta u=f \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega$$

and

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1) For isoparametric elements

$$-\Delta u+u=f \text{ in } \Omega, \quad \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega$$

on a smooth or convex polygonal domain Ω. Let

T_h——triangulation with locally uniform meshes/piecewise uniform meshes,

z——nodal points of T_h having positive distance from $\partial\Omega$/the corner points of $\partial\Omega$/the interior vextex p of the macro triangulation (see [1]),

u^h——linear finite element solution of u,

$i_h u$——piecewise linear interpolant of u,

R_h and R'_h——remainders in the following error expansions:

$$(u^h-u)(z)=h^2 e(z)+R_h(z),$$

$$\partial_s(u^h-i_h u)(z)=h^2\,\partial_s e(z)+R'_h(z).$$

We will see that R_h and R'_h are of higher order, though the order will depend on the smoothness of u and the uniformity of T_h. We assemble some results as follows. (see [1]).

T_h	R_h	R'_h
uniform	$O(h^4)\,\|u\|_{4+s,\infty}$	$O(h^4)\,\|u\|_{5+s,\infty}$
piecewise uniform	$O\left(h^4 \ln \frac{1}{h}\right)\|u\|_{4+s,\infty}$	$O(h^3)\,\|u\|_{4+s,\infty}$
	$O\left(h^3 \ln \frac{1}{h}\right)\|u\|_{3,\infty}$	
	$o(h^2)$ if $u\in W^{3,2+s}(\Omega_0)\cap H^2(\Omega)$	$o(h^2)$ if $u\in W^{4,2+s}(\Omega_0)\cap H^2(\Omega)$
locally uniform	$O\left(h^3 \ln \frac{1}{h}\right)\|u\|_{3,\infty}$	$O(h^3)\,\|u\|_{4+s,\infty}$

where $s>0$, $\Omega_0\subset\Omega$. Notice that even at the point p where the triangulation is only regular in the usual sense, the known error behavior

$$(u^h-u)(p)=O\left(h^2 \ln \frac{1}{h}\right)$$

may also be improved to

$$\frac{1}{3}(4u^{h/2}-u^h)(p)=u(p)+O(h^3).$$

So the Richardson extrapolation increases the accuracy in almost all cases.

We will also see in the second part that an extended expansion of the form

$$(u^h-u)(z)=h^2 e^{(1)}(z)+h^4 e^{(2)}(z)+O(h^4)$$

even holds for interior nodal points z.

Our conclusion is that the finite element method fits Richardson extrapolation well.

§ 2. Expansion of Bramble Functional and the Intergral Error Version

In section 3 the eigenvalue error will be reduced to some integral errors, which