THE STABILITY ANALYSIS OF THE SOLUTIONS OF INVERSE EIGENVALUE PROBLEMS*

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Abstract

This paper gives perturbation bounds of some solutions of the classical additive and multiplicative inverse eigenvalue problems for real symmetric matrices.

§ 1. Problems and Main Results

Throughout this paper we use the following notation. $S\mathbb{R}_0^{n\times n}$ is the set of all $n\times n$ real symmetric matrices with zero diagonal elements, and $S\mathbb{R}_1^{n\times n}$ the set of all $n\times n$ real symmetric matrices with unit diagonal elements. $\mathbb{R}^n$ denotes the set of all n-dimensional real column vectors. The norm $\|\ \|_1$ stands for both the vector 1-norm and the matrix 1-norm. The superscript T is for transpose. For an arbitrary $n\times n$ real symmetric matrix A with eigenvalues $\lambda_1\geqslant\cdots\geqslant\lambda_n$, the symbol $\mu(A)$ denotes the vector $(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$.

The following are the most common inverse eigenvalue problems:

Problem $\boldsymbol{A}(A, \lambda)$. Given $A=(a_{ij})\in S\mathbb{R}_0^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$, find $c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n$ such that the eigenvalues of $A+\mathrm{diag}(c_1, \cdots, c_n)$ are $\lambda_1, \cdots, \lambda_n$.

Problem $\boldsymbol{M}(A, \lambda)$. Given a positive definite matrix $A=(a_{ij})\in S\mathbb{R}_1^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$ with $\lambda_i>0 (i=1, \cdots, n)$, find $c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n$ such that the eigenvalues of $\mathrm{diag}(c_1, \cdots, c_n)A$ are $\lambda_1, \cdots, \lambda_n$.

Problem $\boldsymbol{A}$ is the classical additive inverse eigenvalue problem and Problem $\boldsymbol{M}$ the multiplicative inverse eigenvalue problem. The solubility and the numbers of solutions $c\in\mathbb{R}^n$ as well as numerical methods for Problem $\boldsymbol{A}$ and Problem $\boldsymbol{M}$ have been studied (see [1], [2], [6] and the references contained therein). Nevertheless, to the best of the author's knowledge, the stability analysis of the solutions of Problem $\boldsymbol{A}$ and Problem $\boldsymbol{M}$ is not yet treated, and it is the subject of this paper.

Let

$$g_j=\sum_{k}|a_{jk}|, \quad j=1, \cdots, n \tag{1.1}$$

and

$$\mathscr{D}_\varepsilon=\{c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n: \lambda_1+\varepsilon\geqslant c_1\geqslant c_2\geqslant\cdots\geqslant c_n\geqslant\lambda_n-\varepsilon\},$$

where $\varepsilon>0$ for Problem $\boldsymbol{A}$, and $\lambda_n>\varepsilon>0$ for Problem $\boldsymbol{M}$. The following theorems have been proved by Hadeler[2].

Theorem H-1. *If $A=(a_{ij})\in S\mathbb{R}_0^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$ satisfy $\lambda_1>\lambda_2>\cdots>\lambda_n$ and*

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$$\lambda_j-\lambda_{j+1}>2\max\{g_j,\ g_{j+1}\},\quad j=1,\ \cdots,\ n-1,$$

then there exists a unique solution $c=(c_1,\ \cdots,\ c_n)^T$ *for Problem* $\boldsymbol{A}(A,\ \lambda)$ *in* $\mathscr{D}_s$, *and*

$$|c_j-\lambda_j|\le g_j,\quad j=1,\ \cdots,\ n. \tag{1.2}$$

Theorem H-2. *If* $A=(a_{ij})\in S\mathbb{R}_1^{n\times n}$ *and* $\lambda=(\lambda_1,\cdots,\ \lambda_n)^T\in\mathbb{R}^n$ *satisfy* $\lambda_1>\lambda_2>\cdots>\lambda_n>0$ *and*

$$\lambda_j-\lambda_{j+1}>2\lambda_1\max\{g_j,\ g_{j+1}\},\quad j=1,\ \cdots,\ n-1,$$

then there exists a unique solution $c=(c_1,\ \cdots,\ c_n)^T$ *for Problem* $\boldsymbol{M}(A,\ \lambda)$ *in* $\mathscr{D}_s$, *and*

$$|c_j-\lambda_j|\le\lambda_1 g_j,\quad j=1,\ \cdots,\ n. \tag{1.3}$$

On the basis of Hadeler's theorems we shall prove the following results.

Theorem 1. *Let* $A=(a_{ij}),\ \tilde{A}=(\tilde{a}_{ij})\in S\mathbb{R}_0^{n\times n},\ \lambda=(\lambda_1,\ \cdots,\ \lambda_n)^T,\ \tilde{\lambda}=(\tilde{\lambda}_1,\ \cdots,\ \tilde{\lambda}_n)^T\in\mathbb{R}^n$. *Assume that*

$$\lambda_j-\lambda_{j+1}>2\max\{g_j,\ g_{j+1}\},\ \tilde{\lambda}_j-\tilde{\lambda}_{j+1}>2\max\{\tilde{g}_j,\ \tilde{g}_{j+1}\},\quad j=1,\ \cdots,\ n-1, \tag{1.4}$$

where g_j *is defined by* (1.1), *and*

$$\tilde{g}_j=\sum_{k\ne j}|\tilde{a}_{jk}|,\quad j=1,\ \cdots,\ n. \tag{1.5}$$

Suppose that $c=(c_1,\ \cdots,\ c_n)^T\in\mathscr{D}_s$ *and* $\tilde{c}=(\tilde{c}_1,\ \cdots,\ \tilde{c}_n)^T\in\tilde{\mathscr{D}}_s$ *are the solutions of Problem* $\boldsymbol{A}(A,\ \lambda)$ *and Problem* $\boldsymbol{A}(\tilde{A},\ \tilde{\lambda})$, *respectively, where*

$$\tilde{\mathscr{D}}_s=\{c=(c_1,\ \cdots,\ c_n)^T\in\mathbb{R}^n:\ \tilde{\lambda}_1+s\ge c_1\ge c_2\ge\cdots\ge c_n\ge\tilde{\lambda}_n-s\},\quad s>0.$$

Then

$$\|\tilde{c}-c\|_1<\frac{D}{\delta_A}(2\|\tilde{a}-a\|_1+\|\tilde{\lambda}-\lambda\|_1), \tag{1.6}$$

where

$$a=(a_{12},\ \cdots,\ a_{1n},\ a_{23},\ \cdots,\ a_{2n},\ \cdots,\ a_{n-1,n})^T, \tag{1.7}$$

$$\tilde{a}=(\tilde{a}_{12},\ \cdots,\ \tilde{a}_{1n},\ \tilde{a}_{23},\ \cdots,\ \tilde{a}_{2n},\ \cdots,\ \tilde{a}_{n-1,n})^T, \tag{1.8}$$

$$D=\max_{1\le j\le n-1}\max\{\lambda_j-\lambda_{j+1},\ \tilde{\lambda}_j-\tilde{\lambda}_{j+1}\}, \tag{1.9}$$

$$\delta_A=\min_i\delta_i'$$

and

$$\delta_i'=\min\{\min_{j\ne i}|\lambda_j-\lambda_i|-2g_i,\ \min_{j\ne i}|\tilde{\lambda}_j-\tilde{\lambda}_i|-2\tilde{g}_i\},\quad i=1,\ \cdots,\ n.$$

Theorem 2. *Let* $A=(a_{ij}),\ \tilde{A}=(\tilde{a}_{ij})\in S\mathbb{R}_1^{n\times n},\ \lambda=(\lambda_1,\ \cdots,\ \lambda_n)^T,\ \lambda_1>\cdots>\lambda_n>0,\ \tilde{\lambda}=(\tilde{\lambda}_1,\ \cdots,\ \tilde{\lambda}_n)^T,\ \tilde{\lambda}_1>\cdots>\tilde{\lambda}_n>0$. *Assume that*

$$\lambda_j-\lambda_{j+1}>2\lambda_1^*\max\{g_j,\ g_{j+1}\},\ \tilde{\lambda}_j-\tilde{\lambda}_{j+1}>2\lambda_1^*\max\{\tilde{g}_j,\ \tilde{g}_{j+1}\},\quad j=1,\ \cdots,\ n-1, \tag{1.10}$$

where g_j *and* $\tilde{g}_j$ *are defined by* (1.1) *and* (1.5), *respectively, and*

$$\lambda_1^*=\max\{\lambda_1,\ \tilde{\lambda}_1\}.$$

Suppose that $c=(c_1,\ \cdots,\ c_n)^T\in\mathscr{D}_s$ *and* $\tilde{c}=(\tilde{c}_1,\ \cdots,\ \tilde{c}_n)^T\in\tilde{\mathscr{D}}_s(\tilde{\lambda}_n>s>0)$ *are the solutions of Problem* $\boldsymbol{M}(A,\ \lambda)$ *and Problem* $\boldsymbol{M}(\tilde{A},\ \tilde{\lambda})$, *respectively. Then*

$$\|\tilde{c}-c\|_1<\frac{\lambda_1^*D}{\lambda_n^*\delta_M}(2\lambda_1^*\|\tilde{a}-a\|_1+\|\tilde{\lambda}-\lambda\|_1), \tag{1.11}$$

where a, $\tilde{a}$ *and* D *are defined by* (1.7)—(1.9), *and*

$$\lambda_n^*=\min\{\lambda_n,\ \tilde{\lambda}_n\},$$