ON MODIFIED HERMITE-FEJÉR INTERPOLATION OMITTING DERIVATIVES*1)

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§ 1. Introduction

Let us consider the Hermite-Fejér interpolation

$$H_n(f,x)=\sum_{k=1}^{n} f(x_k)h_{kn}(x), \qquad (1.1)$$

on the interval $[-1, 1]$ for a function $f\in C\,[-1, 1]$ where

$$-1\leqslant x_{nn}<\cdots<x_{1n}\leqslant 1,\quad n=1, 2, \cdots,$$

$$W_n(x)=\prod_{k=1}^{n}(x-x_{kn}),$$

$$l_{kn}(x)=W_n(x)/[W'_n(x_{kn})(x-x_{kn})],\quad k=1, \cdots, n,$$

$$h_{kn}(x)=[1-W''_n(x_{kn})(x-x_{kn})/W'_n(x_{kn})]l^2_{kn}(x),\quad k=1, \cdots, n.$$

It is well-known that for zeros of Chebyshev polynomial $T_n(x)$

$$x_{kn}=\cos\theta_{kn}=\cos(2k-1)\pi/(2n),\quad k=1, \cdots, n, \qquad (1.2)$$

according to a classical result of L. Fejér[1] $H_n(f, x)$ converges uniformly to $f(x)$. In 1960, P. Turán suggested that perhaps omission of derivatives at a "few" exceptional points would not damage the convergence property of the modified Hermite-Fejér polynomial $H^*_{\mu(n)}(f, x)$ with the nodes (1.2). In [2], P. Turán proved that $H^*_{\mu(n)}(f,x)$ does not converge uniformly in general. Later, K. Kumar and K. K. Mathur[3] considered the following question:

Is there any matrix of nodes for which the modified Hermite-Fejér interpolation $H^*_{\mu(n)}(f, x)$ given by

$$H^*_{\mu(n)}(f, x)=H_n(f, x)+(x-x_\mu)l^2_\mu(x)\,W'^2_n(x_\mu)\sum_{k=1}^{n} f(x_k)\frac{W''_n(x_k)}{W'^3_n(x_k)}, \qquad (1.3)$$

satisfying the properties

$$H^*_{\mu(n)}(f, x_k)=f(x_k),\quad k=1, \cdots, n,$$

$$H^{*\prime}_{\mu(n)}(f, x_k)=0,\quad 1\leqslant k\leqslant n,\ k\neq 0,$$

converges uniformly to every $f\in C\,[-1, 1]$. They claimed an affirmative answer for the interpolation $H^*_{\mu(n)}(f, x)$ constructed on the point-systems

$$\{\cos(2k-1)\pi/(2n+1)\}_{k=1}^{n+1}, \qquad (1.4)$$

$$\{\cos 2k\pi/(2n+1)\}_{k=0}^{n}, \qquad (1.4)'$$

$$\{\cos(k-1)\pi/(n-1)\}_{k=0}^{n}. \qquad (1.5)$$

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But, their result was incorrect. In fact, even $H^*_{1(n)}(f_0, 1)$ with the nodes $\{\cos(2k-1)\pi/(2n+1)\}_{k=1}^{n+1}$ does not converge to $f_0(1)$, where $f_0(x)=x$.

On the other hand, P. Turán[2] proved thet uniform convergence of $H^*_{\mu(n)}(f, x)$ with the nodes (1.2) in $[-1, 1]$ holds if and only if

$$\int_{-1}^{1}\frac{xf(x)}{\sqrt{1-x^2}}dx=0. \tag{1.6}$$

Condition (1.6) is related to f. In the present paper the author considers the following question:

What are the necessary and sufficient conditions which ensure that the uniform convergence of $H^*_{\mu(n)}(f, x)$ still holds for every $f\in C[-1, 1]$ when a derivative out of the points (1.4) and (1.5) is omitted.

§ 2. Main Result

Theorem 2.1. *For the interpolation $H^*_{\mu(n)}(f, x)$ constructed on the pointsystem (1.4) uniform convergence to every $f\in C[-1, 1]$ holds if and only if*

$$n-\mu(n)=O(1). \tag{2.1}$$

Proof. Denote by $H_n(f, x)$ the Hermite-Fejér operator based on the nodes $\{\cos(2k-1)\pi/(2n+1)\}_{k=1}^{n+1}$. From (1.3) we have

$$H^*_{\mu(n)}(f, x)=H_n(f, x)+J_n(x), \tag{2.2}$$

$$J_n(x)=\frac{(1+x)^2P_n^{(-\frac{1}{2}, \frac{1}{2})}(x))^2}{x-x_\mu}\Big[\frac{2}{(2n+1)^2}\sum_{k=1}^{n}\frac{f(x_k)}{1+x_k}-\frac{2n(n+1)}{3(2n+1)^2}f(-1)\Big],$$

where $P_n^{(-\frac{1}{2}, \frac{1}{2})}(x)=\cos(2n+1)\frac{\theta}{2}\Big/\cos\frac{\theta}{2}(x=\cos\theta)$. To prove (2.1) is necessary, suppose that $H^*_{\mu(n)}(f, x)$ converges uniformly to every $f\in C[-1, 1]$. On using Theorem 1 of [4], i.e., $\lim\limits_{n\to\infty}H_n(f, x)=f(x)$ uniformly for every $f\in C[-1, 1]$, we have that for every $f\in C[-1, 1]$

$$\lim_{n\to\infty}J_n(x)=0$$

holds uniformly. Particularly, when $x^*=\cos\theta^*$, $\theta^*=\theta_\mu-\pi/[2(2n+1)]$ and $f(x)=\Omega(1+x)$ where $\Omega(x)$ satisfies the following conditions:

$$\begin{aligned}&\text{(i) } \Omega(x)\in C[0, 2] \text{ and } \Omega(x) \text{ is nondecreasing,}\\ &\text{(ii) } \Omega(x)\geqslant 0\ (x\geqslant 0) \text{ and } \Omega(0)=0;\end{aligned} \tag{2.3}$$

noting that

$$\sum_{k=1}^{n}1/(1+x_k)=n(n+1)/3,$$

we have that

$$\lim_{n\to\infty}J_n(x^*)=\lim_{n\to\infty}\Big\{\frac{\cos^2\theta^*/2}{\sin\frac{1}{2}(\theta_\mu-\theta^*)\sin\frac{1}{2}(\theta_\mu+\theta^*)}\cdot\frac{2}{(2n+1)^2}\sum_{k=1}^{n}\frac{\Omega(1+x_k)}{1+x_k}\Big\}=0 \tag{2.4}$$

holds. From tne monotonicity of $\Omega(x)$ we obtain