NUMERICAL SOLUTION FOR THE STEFAN PROBLEM WITH CERTAIN SINGULARITIES*[1]

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§ 1. Introduction

In this paper the Stefan problem refers to the heat flow problem of the materials which undergo phase change. As is well-known, it is an important example of the free boundary problem. In recent years, there have been a great number of numerical methods for the Stefan problem: difference methods in which one establishes difference schemes for the given original problem; enthalpy methods in which one gets rid of the free boundary through introducing enthalpy, a function of temperature; variational methods, finite element methods, etc. For the survey of these methods, we refer the readers to [1], [2].

However, most of these numerical methods do not deal with the discontinuity of the temperature at the origin $(0, 0)$ or at some point $(b, 0)$ $(b>0)$.

Due to the discontinuity, the derivatives of temperature with respect to t and x must be very large in the neighbourhood of the point of discontinuity. R. Bonnet and P. Jamet[3] suggested a third-order method which permits the discontinuity in the t direction. They computed the Stefan problem with initial discontinuity. Later, according to the idea of [3], Chin Hsien Li[4] suggested another method which is applicable to the problem whose free boundary is in an implicit form. The result given in [3] is good if t is not very small, but near the point of discontinuity, the error is not very small. This is shown by our computation given in § 7.

One of the authors has developed the singularity-separating difference method, by which very accurate numerical results on discontinuous solutions of quasilinear hyperbolic systems have been obtained[5]. Moreover, based on the analytical property of solution of flow in the region where a shock wave passes through a "strong explosion" center, a quite effectual method for this problem has been proposed in [6] and very good results have been achieved. These experiences show that it is possible to establish a numerical method with high accuracy for the problems with certain singularities by considering the properties of solutions.

A similar idea was proposed by L. Fox in [1]. He pointed out that the combination of numerical and analytical methods is the best way to solve the problem. Especially for the problem with singularity, this kind of method not only avoids the disorder caused by the singularity, but also can clearly describe the state of solution near the singularity.

Apcording to these experiences and ideas, we suggest a new method which could give very accurate results everywhere. The feature of this difference method is that by using the analytical solution given in [7], we introduce a new coordinate system under which the derivatives are not large before the schemes are established. In this paper we give the method and some numerical results, and compare it with other two methods. Error comparison shows that our method is better than the method in [3] and the Galerkin method[8].

§ 2. Formulation of the Problem

Consider the ice-water system

$$c_1(x, t)\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\Big(k_1(x, t)\frac{\partial u_1}{\partial x}\Big), \quad \text{in } \Omega_1, \tag{2.1}$$

$$c_2(x, t)\frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x}\Big(k_2(x, t)\frac{\partial u_2}{\partial x}\Big), \quad \text{in } \Omega_2 \tag{2.2}$$

with boundary conditions

$$u_1(0, t) = g(t), \tag{2.3}$$

$$u_2(Q, t) = h(t), \tag{2.4}$$

free boundary conditions on the phasechange line $x = b + \xi(t)$

$$u_1(b+\xi(t), t) = u_2(b+\xi(t), t) = 0, \tag{2.5}$$

$$\lambda\frac{d\xi}{dt} = k_1(x, t)\frac{\partial u_1}{\partial x}\Big|_{x=b+\xi(t)} - k_2(x, t)\frac{\partial u_2}{\partial x}\Big|_{x=b+\xi(t)}, \tag{2.6}$$

and initial condition

$$\begin{cases} u_1(x, 0) = \mu(x), & 0 < x < b, \tag{2.7} \\ u_2(x, 0) = f(x), & b < x < Q, \tag{2.8} \end{cases}$$

where

$\Omega_1 = \{(x, t): 0 < x < b + \xi(t), t > 0\}$ is the solid state region;

$\Omega_2 = \{(x, t): b + \xi(t) < x < Q, t > 0\}$ is the liquid state region;

and λ is the latent heat of phasechange. Moreover, we make the following assumptions:

Assumption A. $c_i(x, t)$, $k_i(x, t)$ $(i = 1, 2)$, $g(t)$, $h(t)$, $\mu(x)$, $f(x)$ are all continuous functions.

Assumption B. $\mu(b) < 0 < f(b)$ (if $b = 0$, we assume $g(0) < 0 < f(0)$ instead).

We refer to (2.1)—(2.8) as System (SI); when $b = 0$, we refer to (2.1)—(2.6), (2.8) as System (S).

For the sake of later reference, we list some notation here.

$$c_i = c_i(b, 0), \quad k_i = k_i(b, 0), \quad i = 1, 2,$$

$$g = g(0), \quad \mu = \mu(b), \quad f = f(b).$$

§ 3. Systems (CSI) and (CS)

Suppose we have the following equations and conditions:

$$c_1\frac{\partial u_{10}}{\partial t} = k_1\frac{\partial^2 u_{10}}{\partial x^2}, \quad \text{in } \Omega_{10}, \tag{3.1}$$