ON GLOBAL CONVERGENCE AND APPROXIMATE ITERATION OF THE LINEAR APPROXIMATION METHOD FOR SOLVING VARIATIONAL INEQUALITIES*

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Abstract

This paper is concerned with the linear approximation method (i.e. the iterative method in which a sequence of vectors is generated by solving certain linearized subproblems) for solving the variational inequality. The global convergent iterative process is proposed by applying the continuation method, and the related problems are discussed. A convergent result is obtained for the approximation iteration (i.e. the iterative method in which a sequence of vectors is generated by solving certain linearized subproblems approximately).

§ 1. Introduction

Given a subset C of R^n and a mapping f from C into R^n, the variational inequality problem VI(C, f) is to find a vector $x^*\in C$ such that

$$\langle y-x^*, f(x^*)\rangle\geqslant 0, \quad \forall y\in C. \tag{1}$$

An efficient numerical method for solving VI(C, f) is the following iterative scheme.

Algorithm 1. Given $y^k\in C$,

$$y^{k+1}=(1-\alpha_k)y^k+\alpha_k x^{k+1}, \quad \alpha_k\in(0, 1], \tag{2}$$

$$x^{k+1} \text{ solves VI}(C, f^k), \tag{3}$$

where

$$f^k(x)=f(y^k)+A(y^k)(x-y^k)$$

and $A(y^k)$ is an n by n matrix.

We regard Algorithm 1 as a linear approximation method. Included in the family of linear approximation methods are the Newton method, the quasi-Newton method, the SOR method, the linearized Jacobi method and the projection method, etc.

For $\alpha_k\equiv 1$, Rockafellar has established in [3] a convergence theory for Algorithm 1 by the norm-contraction approach, the vector-contraction approach and the monotone approach. His main result by the norm-contraction approach is the following theorem.

Theorem 1. *Assume that*

1) $C\subset R^n$ *is a nonempty closed convex subset;*

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2) $f: C \to R^n$ and $A: R^n \to R^{n\times n}$ *are continuous;*

3) x^* *solves problem* $\mathrm{VI}(C, f)$;

4) *there exists a positive semi-definite metrix* G *such that* $A(x^*) - G$ *is positive semi-definite;*

5) *there exists a neighborhood* N *of* x^* *such that*

$$\|\tilde{G}^{-1}[f(x) - f(y) - A(y)(x-y)]\|_{\tilde{G}} \leqslant b\|x-y\|_{\tilde{G}} \quad \text{for all } x \in R^n,$$

where $b<1$, $\tilde{G} = \frac{1}{2}(G+G^T)$ *and* $\|\cdot\|_{\tilde{G}}$ *is defined as*

$$\|x\|_{\tilde{G}} = (x^T \tilde{G} x)^{\frac{1}{2}} \quad \text{for all } x \in R^n.$$

Then provided that the initial vector y^0 *is chosen in a suitable neighborhood of* x^*, *the sequence* $\{y^k\}$ *generated by Algorithm* 1 *with* $\alpha_k \equiv 1$ *is well defined and converges to the solution* x^*. *Moreover, there is an* $r \in (0, 1)$ *such that*

$$\|y^{k+1} - x^*\|_{\tilde{G}} \leqslant r \cdot \|y^k - x^*\|_{\tilde{G}} \quad \text{for } k \geqslant 0. \tag{4}$$

By using the results of Theorem 1, especially (4), one can prove the following corollary without any difficulty.

Corollary 1. Assume that hypotheses 1, 2, 3, 4, 5 in Theorem 1 hold, and assume that

6) $1 \geqslant \alpha_k \geqslant \alpha > 0, \quad \forall k.$

Then provided that the initial vector y^0 is chosen in a suitable neighborhood of x^*, the sequence $\{y^k\}$ generated by Algorithm 1 is well defined and converges to the solution x^*. Moreover, there is an $r \in (0, 1)$ such that

$$\|y^{k+1} - x^*\|_{\tilde{G}} \leqslant r\|y^k - x^*\|_{\tilde{G}} \quad \text{for } k \geqslant 0.$$

This paper will deal with two problems about Algorithm 1.

1) Generally, Algorithm 1 is locally convergent. Is there some method to extend it to a global convergent algorithm, or, alternatively, is there some procedure to obtain a sufficiently close starting point y^0?

2) When the calculation of (3) is non-exact (i.e. x^{k+1} is an approximation to the solution, not the solution, of $\mathrm{VI}(C, f^k)$, does Algorithm 1 converge and on what condition does it converge?

A question similar to problem 2) was proposed by Rockafellar[4] to the Penalty-Duality method, which is devised for solving variational inequalities, but he pointed out that the answer had been not obtained.

For the first question, applying the continuation method[2] to Algorithm 1, we obtained a global convergent algorithm. The convergence is proved and the related problems are discussed. By adding a very mild (but essential) condition, we obtain a convergent result for the second question.

In the following two sections, we shall deal with the two problems respectively.

§ 2. Global Convergent Iteration

Let a homotopy $H(\cdot, \cdot): C \times [0, 1] \subset R^n \times R^1 \to R^n$. We want to solve problem $\mathrm{VI}(C, H(x, 1))$, and the solution of $\mathrm{VI}(C, H(x, 0))$ is given. Assume $x(t)$ continuously depends on t.