THE USE OF MAXIMAL MONOTONE OPERATORS IN THE NUMERICAL ANALYSIS OF VARIATIONAL INEQUALITIES AND FREE BOUNDARY PROBLEMS*

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Abstract

In this work we use maximal monotone operators theory for solving some variational inequalities and some free boundary problems. This utilization, already adopted at theoretical level for giving existence, uniqueness and regularity theorems ([1], [6], [10]) in this field, was ignored until now in numerical methods. We obtain in this way a direct computation method extremely simple and general enough as we shall prove in concrete examples.

Some Second Order Problem

§ 1. Introduction

We adopt usual notations in the field of variational inequalities ([5], [8]). Let V be a real Sobolev space,

K a closed convex set in V,

$a(\cdot, \cdot)$ a bilinear, continuous, V-elliptic form on $V\times V$:

$$|a(u, v)| \leqslant M\|u\|\|v\|, \quad \forall u, v\in V,$$

$$a(u, u)\geqslant \alpha\|u\|^2, \alpha>0, \quad \forall u\in V,$$

$\langle\cdot, \cdot\rangle$ the duality pairing between V' and V,

χ: $V\to R$ a proper convex function, lower-semicontinuous, on V, having K as effective domain,

$f\in V'$ an assigned function, which we suppose belongs to $L^2(\Omega)$ in numerical examples.

We consider the following variational inequality:

$$\text{to find } u\in V: a(u, v-u)+\chi(v)-\chi(u)\geqslant\langle f, v-u\rangle, \forall v\in V. \tag{1.1}$$

It is well known that (1.1) has a unique solution ([8]).

We recall that a vector $\mu(u)$ is said to be a subgradient of χ at a point u, if $\mu(u)$ satisfies the "subgradient inequality":

$$\chi(v)-\chi(u)\geqslant\langle\mu(u), v-u\rangle, \forall v\in V. \tag{1.2}$$

The subdifferential $\partial\chi(u)$, a multivalued mapping from V to $2^{V'}$, is the set of all subgradients ([7]). We suppose $\partial\chi(v)\neq\emptyset$, $\forall v\in K$; (0, 0) belongs to the graph of $\partial\chi$. If we consider the operator A: $V\to V'$ associated with $a(\cdot, \cdot)$:

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$$\langle Au, v\rangle = a(u, v), \tag{1.3}$$

we obtain from the theory of maximal monotone operators ([1], [11]) that A and $\partial\chi$ as well as $A+\partial\chi$ are maximal monotone operators. Besides, $A+\partial\chi$ is coercive. Thus the problem:

$$\{u, \mu(u)\}\in V\times V': \langle Au+\mu(u), v\rangle = \langle f, v\rangle, \quad \forall v\in V \tag{1.4}$$

has a solution ([11], Corollary, p. 120) and only one solution. In fact, let u_1, u_2 be two solutions; by setting $u=u_1$, $v=u_1-u_2$ in (1.4) and then $u=u_2$, $v=u_2-u_1$, we obtain respectively

$$\langle Au_1+\mu(u_1), u_1-u_2\rangle = \langle f, u_1-u_2\rangle, \tag{1.5}$$

$$\langle Au_2+\mu(u_2), u_2-u_1\rangle = \langle f, u_2-u_1\rangle. \tag{1.6}$$

By adding (1.5), (1.6) and by using V-ellipticity of A and monotonicity of $\partial\chi$ we have

$$0\leqslant\alpha\|u_1-u_2\|^2\leqslant\langle A(u_1-u_2), u_1-u_2\rangle+\langle\mu(u_1)-\mu(u_2), u_1-u_2\rangle = 0 \tag{1.7}$$

which proves that $u_1=u_2$. $\mu(u)$ is a section of $\partial\chi$. (1.4) is equivalent to (1.1). The same type of equation appears in some free boundary problems, for example in Stefan's problem.

§ 2. Two Obstacles Problem

Let

Ω be an open, bounded set in R^n with boundary $\partial\Omega=\Gamma$ regular enough. Ω is a convex polygon in R^2, in numerical examples,

$V=H_0^1(\Omega)$,

$a(u, v)=\int_\Omega \operatorname{grad} u\times\operatorname{grad} v\cdot dx$, $Au=-\Delta u$,

$K_\alpha^\beta=\{v: v\in V/\beta\geqslant v\geqslant\alpha$ a.e. in $\Omega\}$. α and β are assigned functions such that $\beta|_\Gamma\geqslant 0\geqslant\alpha|_\Gamma$; we can suppose $\alpha, \beta\in C^0(\overline{\Omega})$ and $\alpha=0$ by translation.

We consider the variational inequality ([5], Chapter II, Section 6):

$$u\in K_\alpha^\beta: a(u, v-u)\geqslant\langle f, v-u\rangle, \quad \forall v\in K_\alpha^\beta, \tag{2.1}$$

and consequently have as "indicator" function of K_α^β

$$\chi(v)\begin{cases} =0 & \text{if } v\in K_\alpha^\beta, \\ =+\infty & \text{if } v\notin K_\alpha^\beta \end{cases} \tag{2.2}$$

and then like subdifferential:

$$\partial_\chi(v)\begin{cases} =\emptyset & \text{if } v\notin K_\alpha^\beta, \\ =]-\infty,0] & \text{if } v=\alpha, \\ =0 & \text{if } \beta>v>\alpha, \\ =[0, +\infty[& \text{if } v=\beta. \end{cases}$$

(2.1) is equivalent to

$$u\in V: a(u, v-u)+\chi(v)-\chi(u)\geqslant\langle f, v-u\rangle, \quad \forall v\in V, \tag{2.4}$$

$$\{u, \mu(u)\}\in V\times V': a(u, v)+\langle\mu(u), v\rangle=\langle f, v\rangle, \quad \forall v\in V. \tag{2.5}$$

We must determine, besides u, a measure $\mu(u)=\mu^+-\mu^-$ satisfying (2.5).