THE EXACT ESTIMATION OF THE HERMITE-FEJÉR INTERPOLATION*1)

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Abstract

The exact pointwise estimation of the Hermite-Fejér interpolation process based on the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(-1<\alpha, \beta<0)$ is given. The method employed is useful for other extended H-F interpolations also.

§ 1. Introduction and Result

Let $f\in C[-1, 1]$ and (with $x_k=x_{kn}^{(\alpha,\beta)}$, $k=1, 2, \cdots, n$)

$$-1<x_{nn}^{(\alpha,\beta)}<\cdots<x_{2n}^{(\alpha,\beta)}<x_{1n}^{(\alpha,\beta)}<1$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta)}(x)=(1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}], \quad \alpha, \beta>-1.$$

Define the Hermite-Fejér interpolation by

$$H_n^{(\alpha,\beta)}(f, x)=\sum_{k=1}^{n} f(x_k)v_k(x)l_k^2(x), \tag{1.1}$$

where

$$l_k(x)=l_{kn}^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x)/[P_n'^{(\alpha,\beta)}(x_k)(x-x_k)],$$

$$v_k(x)=v_{kn}^{(\alpha,\beta)}(x)=\{1-x[\alpha-\beta+(\alpha+\beta+2)x_k]+(\alpha-\beta)x_k+(\alpha+\beta+1)x_k^2\}/(1-x_k^2),$$

$$k=1, \cdots, n.$$

Denote by $\omega(t)$ the given modulus of continuity and $H_\omega=\{f;\ \omega(f, t)\leq\omega(t)\}$, where $\omega(f, t)$ is the modulus of continuity of f. In what follows, c, c_1, $c_2>0$ or the sign "0" will always denote different constants that are independent of f, n and x but dependent on α and β. The sign "A∼B" means that there exist constants c_1 and c_2 such that

$$c_1A\leq B\leq c_2A.$$

In recent years there has been a great amoumt of research concerning the degree of approximation by interpolation process (1.1). P. Vertesi[1] proved that

$$|H_n^{(\alpha,\beta)}(f, x)-f(x)|=O(1)\sum_{i=1}^{n}\left[\omega\left(f, \frac{i\sqrt{1-x^2}}{n}\right)+\omega\left(f, \frac{i^2|x|}{n^2}\right)\right]i^{2r-1},$$

where $r=\max\left(\alpha, \beta, -\frac{1}{2}\right)$. Many authors (see [2]) investigated the special cases $H_n^{(-\frac{1}{2}, -\frac{1}{2})}(f, x)$ and $H_n^{(0,0)}(f, x)$. Here we cite the following works only:

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$$|H_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)-f(x)|=O\Big(\frac{1}{n}\Big)\sum_{k=1}^{n}\omega\Big(f,\ \frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\Big)^{[3]},$$

$$|H_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)-f(x)|\le C\Big\{\omega(f,\ |\theta-\theta_{k_0}|^2|x|+|\theta-\theta_{k_0}|\sqrt{1-x^2})$$
$$+C\,\frac{T_n^2(x)}{n}\int_{\frac{1}{n}}^{1}\omega(f,\ t^2|x|+t\sqrt{1-x^2})t^{-2}\,dt\Big\}^{[4]},$$

where $\theta=\arccos x$ and $\theta_{k_0}=(2k_0-1)\pi/(2n)$ satisfy the inequality $|\theta-\theta_{k_0}|\le\pi/(2n)$,

$$|H_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)-f(x)|=O\Big(\frac{1}{n}\Big)\sum_{k=1}^{n}\omega\Big(f,\ \frac{\sqrt{1-x^2}|T_n(x)|}{k}+\frac{|x|\,|T_n(x)|}{k^2}\Big)^{[5]}.$$

The improvement of the degree of approximation naturally gives rises to the question of whether the above results are exact? This is a problem raised by Prof. Shen Xie-chang[2]. The purpose of the paper is to answer the above question and to give the exact pointwise degree of $H_n^{(\alpha,\beta)}(f,x)$ $(-1<\alpha,\ \beta\le 0)$. Our method is useful for finding the exact degree of other H-F type interpolations.

Our main result is the following

Theorem. *For every $f\in C[-1,1]$ and $-1<\alpha,\ \beta\le 0$, we have*

$$\sup_{f\in H_\omega}|H_n^{(\alpha,\beta)}(f,x)-f(x)|\sim\begin{cases}L_\alpha(x), & if\ \ 0\le x\le 1,\\ L_\beta(x), & if\ \ -1\le x\le 0,\end{cases}\tag{1.2}$$

where

$$L_t(x)=\begin{cases}\omega(|x-x_j|)v_j(x)+K_t(x), & if\ \ -1<t\le-\dfrac{1}{2},\\[2mm] \omega(|x-x_j|)v_j(x)+\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I_t(x)+t\cdot J_t(x), & if\ -\dfrac{1}{2}<t\le 0,\end{cases}$$

and

$$K_t(x)=n(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\Big(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\Big)\Big(\sqrt{1-x^2}+\frac{i}{n}\Big)^{1+2t}\cdot i^{-2},\quad -1<t\le-\frac{1}{2},$$

$$I_t(x)=n(1-x^2)(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\Big(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\Big)\Big(\sqrt{1-x^2}+\frac{i}{n}\Big)^{-1+2t}\cdot i^{-2},$$
$$-\frac{1}{2}<t\le 0,$$

$$J_t(x)=n^{-2t}(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\Big(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\Big)\cdot i^{-1+2t},\quad -\frac{1}{2}<t\le 0,$$

x_j satisfies $|x-x_j|=\min\limits_{1\le k\le n}|x-x_k|$.

§ 2. Lemmas

In order to prove the above Theorem we need some known results and the following lemmas. Let $x=\cos\theta$, $x_k=\cos\theta_k$,

$$|P_n'^{(\alpha,\beta)}(x_k)|\sim k^{-\alpha-\frac{3}{2}}n^{\alpha+2},\quad 0<\theta_k\le\frac{\pi}{2},\tag{2.1}$$

$$\theta_k\sim n^{-1}\{k\pi+O(1)\},\tag{2.2}$$