ON THE STRUCTURE OF BÉZIER NETS*1)

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Abstract

The distance between two neighbouring multivariate Bézier nets $G_n^m\hat{f}$ and $G_n^{m+1}\hat{f}$ is proved to be $O(m^{-2})$ in this paper. As a consequence, the sequence of Bézier nets is uniformly convergent with the optimal approximation order $O(m^{-1})$. Furthermore, the structures of Bézier nets are explored by investigating how the piecewise linear surface $G_n^m\hat{f}$ tends to the Bézier surface of C^∞.

§ 1. Introduction

It is well known that the Bézier surfaces have been established as a mathematical basis of many CAD systems. "Bernstein–Bézier approximations have recently become very popular and no fewer than 1/4 of the titles at this symposium contain Bézier's name"[1]. The Bézier nets associated with Bézier surface are a very useful tool in exploring the Bézier surface. Many properties of Bézier nets have been explored, such as the limit of Bézier nets, the variation diminishing properties, the convexity preservation. The approximation order of Bézier nets in the univariate case has been shown in [5] to be $O(1/m)$. [5] also shows the relationship between the convergence of Bézier nets and the approximation of Bernstein–Bézier polynomials.

In this paper we are concerned with the Bézier nets and Bézier surface on a triangle in R^n for the bivariate and multivariate cases. Starting from the point of view that the Bézier nets are obtained by successive piecewise linear interpolation, we prove that the distance between the neighbouring Bézier nets $G_n^m\hat{f}$ and $G_n^{m+1}\hat{f}$ is $O(1/m^2)$. As an immediate consequence, the sequence of Bézier nets is uniformly convergent with the approximation order $O(1/m)$. In searching for the representation of the limit of Bézier nets, we show how the piecewise liner surface $G_n^m\hat{f}$ tends to the Bézier surface of C^∞. Therefore the structures of Bézier nets are explored more clearly.

Now, we introduce some notations used in this paper.

The domain of the Bézier surface is a triangle T with three vertices $T_i=(\hat{x}_i, \hat{y}_i)$, $i=1, 2, 3$.

Every point P in T is identified with its barycentric coordinates, i.e.,

$$P=(x, y)=uT_1+vT_2+wT_3,$$
$$u\geqslant 0,\ v\geqslant 0,\ w\geqslant 0,$$
$$u+v+w=1.$$

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Let f be a function defined on T, and $f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)$ be denoted by $f_{i,j,k}$ for $i+j+k=n$.

$E_k(k=1, 2, 3)$ are shift operators, defined by

$$E_1 f_{i,j,k}=f_{i+1,j,k},$$
$$E_2 f_{i,j,k}=f_{i,j+1,k},$$
$$E_3 f_{i,j,k}=f_{i,j,k+1},$$

and $E_{-k}=(E_k)^{-1}$, $k=1, 2, 3$.

Denote by $\Delta_{k,s}^{r,t}(1\leqslant k, s, r, t\leqslant 3)$ the partial difference operator of order 2, i.e.,

$$\Delta_{k,s}^{r,t}=(E_r-E_t)(E_{-k}-E_{-s}).$$

G_n^m is a degree raising operator of order m, defined by

$$G_n^1 f_{i,j,k}=\frac{1}{n+1}(iE_{-1}+jE_{-2}+kE_{-3})f_{i,j,k}$$

for $i+j+k=n+1$ and

$$G_n^m=G_{n+m-1}^1 G_n^{m-1}.$$

Denote by $G_n^m\hat{f}$ the m-th Bézier net for $P=\left(\frac{i}{n+m}, \frac{j}{n+m}, \frac{k}{n+m}\right)\in T$:

$$G_n^m\hat{f}(P)=G_n^m f_{i,j,k}.$$

$B_n(f)$ is a Bernstein polynomial of degree n, i.e.,

$$B_n(f)=\sum_{i+j+k=n} f_{i,j,k}B_{i,j,k}^n(u, v, w)$$

where $\left\{B_{i,j,k}^n=\frac{n!}{i!j!k!}u^i v^j w^k\right\}$ are Bernstein polynomial basis functions.

Throughout this paper we use the maximum norm, i.e.,

$$\|f\|=\max_{P\in T}|f(P)|.$$

§ 2. Main Results

In this section we will present our results on the structure of Bézier nets over a triangle. In order to get an estimate of the distance between $G_n^m\hat{f}$ and $G_n^{m+1}\hat{f}$, we first prove the following identity concerning G_n^m and $\Delta_{k,s}^{r,t}$.

Lemma 1. $G_n^1\Delta_{k,s}^{r,t}-\Delta_{k,s}^{r,t}G_n^1=\frac{1}{n+1}(E_{-k}+E_{-s})\Delta_{k,s}^{r,t}$,

$$G_{n+m-1}^1\Delta_{k,s}^{r,t}G_n^{m-1}-\Delta_{k,s}^{r,t}G_n^m=\frac{1}{n+m}(E_{-k}+E_{-s})\Delta_{k,s}^{r,t}G_n^{m-1}.$$

Proof. It is easy to verify by the definition.

Lemma 2. *For* $1\leqslant k, s, r, t\leqslant 3$, $i_s\geqslant 1$, $i_k\geqslant 1$,

$$|\Delta_{k,s}^{r,t}G_n^m f_{i_1,i_2,i_3}|\leqslant\frac{n(n-1)}{(n+m)(n+m-1)}\max_{\substack{j_1+j_2+j_3=n\\ j_s\geqslant 1, j_k\geqslant 1}}|\Delta_{k,s}^{r,t}f_{j_1,j_2,j_3}|.$$

Proof. Without loss of generality, we let $t=k=1$, $r=s=2$. By Lemma 1, we get