A NEW CONDITION NUMBER OF THE EIGENVALUE AND ITS APPLICATION IN CONTROL THEORY *

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Abstract

Since the present condition numbers of the eigenvalue are not convenient for control systems design, this paper puts forward a new condition number and indicates its application in designing robust control systems.

1. Introduction

A most important tend in the development of control theory is the design of robust systems [1]. However, the system robustness is heavily related to the condition number of the eigenvalue. If we can design a system which has not only satisfactory properties (such as having prescribed eigenvalues), but also a good condition number, then, when the model is exactly equal to the actual system, the system will have a good working properties, and when there exist errors between the model and the system, the system performance will not be too bad. This field has attracted many control theory scholars and engineers [2], [3]. But the existing condition numbers of the eigenvalues are not convenient for designing systems, so the present methods (such as [4], [5]) cannot ensure the condition number of the eigenvalue of the system to be an optimal condition number in actual constraints. This paper proposes a new condition number for solving the problem.

2. Main Result

It is well known that, if $X^{-1}AX = J$, where J is a Jordan matrix, then the condition number of the eigenvalue of matrix A becomes $\|X^{-1}\| \cdot \|X\|$. The most popular condition number is the spectral condition number, $K_2(A) = \sigma_1/\sigma_n$, where σ_n, σ_1 are the smallest singular value and the greatest singular value of matrix X, respectively. But σ_1 and σ_n cannot be determined if the matrix X is not determined. He [6] gives another condition number as follows:

$$K(A) = (E[X_k X_1 X_2 \cdots X_{k-1}])^{-1}$$

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where $X_i (i = 1, \cdots, k)$ are column vectors of X, which are standardized, $E[X_k X_1 \cdots X_{k-1}] = [G(X_1 \cdots X_k)/G(X_1 \cdots X_{k-1})]^{-1}$, G is a Gram determinant, and the subscript k equals ε–rank of the matrix X. Obviously, the definition is too complex to compute. we give another definition.

Definition. *Suppose A is matrix of order n, and $X_i (i = 1, \cdots, n)$ are standardized eigenvectors or generalized eigenvectors of matrix A. $X = (X_1, X_2, \cdots, X_n)$. Define the condition number of the eigenvalue of matrix A as*

$$K(A) = \frac{1}{|X^T X|}.$$

New, we will prove that the definition is reasonable. First, we introduce the following

Lemma [7]. *Suppose that the real coefficient polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with $a_n > 0$ and a_{n-k} being the first negative , and B is the greatest value among the absolute values of the negative coefficients. Then*

$$N = 1 + \sqrt[k]{B/a_n}$$

is an upper bound of the roots of $f(x)$.

Theorem. *Suppose X is a matrix composed of standardized eigenvectors and generalized eigenvectors of A. $X = (X_1, X_2, \cdots, X_n)$. Note $\Delta = \det(X^T X)$. Then*

$$\sigma_n^2 \geq \frac{1}{1 + \frac{1}{\Delta}\left(\frac{n}{n-1}\right)^{n-1}}$$

where $\sqrt{\sigma_n^2}$ is the minimal singular value of X.

Proof . Note $\lambda_i = \sigma_i^2$. Since λ_i are eigenvalues of $X^T X$, so

$$(X^T X) Y = Y \cdot \text{ diag } (\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2), \tag{1}$$

$$\sigma_i^2 \geq \sigma_j^2, \quad i > j, \tag{2}$$

$$\prod_{i=1}^{n} \sigma_i^2 = \prod_{i=1}^{n} \lambda_i = X^T X = \Delta. \tag{3}$$

Since X_i are standardized vectors, so

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma_i^2 = \text{ tr } (X^T X) = n. \tag{4}$$

Note

$$K_1 = \Big\{\sigma_i | \sigma_i \text{ satisfy } (1), (2), (3), (4)\Big\},$$
$$K_2 = \Big\{\sigma_i | \sigma_i \text{ satisfy } (3), (4)\Big\}.$$