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l_2–STABILITY OF DIFFERENCE MODELS FOR HYPERBOLIC INITIAL BOUNDARY VALUE PROBLEMS *

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Abstract

It is showed that, for many commonly used difference models on hyperbolic initial boundary value problems, the necessary and sufficient condition for GKS-stability (in the sense of Definition 3.3 of [1]) is a necessary condition for l_2-stability.

§1. Introduction

Consider the mixed initial boundary value problem

$$\begin{cases} \partial U(x,t)/\partial t = A\partial U(x,t)/\partial x, & x \geq 0,\ t > 0, \\ U(x,0) = f(x), & x \geq 0, \\ U^{\rm I}(0,t) = SU^{\rm II}(0,t) + g(t), & t \geq 0. \end{cases} \tag{1.1}$$

Here A is a constant square matrix, and

$$U(x,t) = (u^{(1)}(x,t), \cdots, U^{(N)}(x,t))^T$$

is a vector function. Furthermore, A is diagonalizable and of the form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{with } A_1 < 0,\ \ A_2 > 0. \tag{1.2}$$

$U^{\rm I}(0,t) = (u^{(1)}, \cdots, u^{(l)})^T$ and $U^{\rm II}(0,t) = (u^{(l+1)}, \cdots, u^{(N)})^T$ correspond to the partition of A , S is a rectangular matrix, and $f(x)$ and $g(t)$ are given functions.

We want to solve the above problem by the general consistent multistep difference model Q:

$$\begin{cases} Q_{-1}U_j^{n+1} = \sum\limits_{\sigma=0}^{s} Q_\sigma U_j^{n-\sigma}, & n \geq s,\ j = 1,2,3,\cdots, \\ U_j^\sigma = f_j^\sigma, \quad 0 \leq \sigma \leq s, & j = -r+1,\ -r+2,\cdots, \\ U_\mu^{n+1} = \sum\limits_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma} + g_\mu^n, & -r+1 \leq \mu \leq 0,\ \ n \geq s. \end{cases} \tag{1.3}$$

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Here $U_j^n \simeq U(jh, n\tau)$ denotes the difference solution, h and τ are mesh width of space step and time step, respectively. The ratio $\tau/h = \lambda$ is a constant.

$$Q_\sigma = \sum_{i=-r}^{p} A_{i\sigma} K^i, \quad KU_j^n = U_{j+1}^n, \quad -1 \le \sigma \le s$$

are difference operators with matrix coefficients, and Q_{-1} is invertible.

$$S_\sigma^{(\mu)} = \sum_{j=0}^{q} C_{j\sigma}^{(\mu)} K^j, \quad -r+1 \le \mu \le 0, \quad -1 \le \sigma \le s$$

are one-sided difference operators. The initial value f_j^σ and boundary value g_μ^n are given $(0 \le \sigma \le s, j \ge -r+1, n \ge s, -r+1 \le \mu \le 0)$. s, p, r, q are nonnegative integers. As usual, we need the following assumptions.

Assumption 1.1. The difference model (1.3) can be solved boundedly for U^{n+1}, i.e., there is a constant $M > 0$ sucy that, for every $G \in l_2(x)$, there is a unique solution $W \in l_2(x)$ of

$$\begin{cases} Q_{-1} W_j = G_j, & j = 1, 2, 3, \cdots, \\ W_\mu - S_{-1}^{(\mu)} W_1 = g_\mu, & -r+1 \le \mu \le 0, \end{cases}$$

With $\|W\|_x^2 \le M(\|G\|_x^2 + h \sum_{\mu=-r+1}^{0} |g_\mu|^2)$. Here $\|W\|_x^2 = \sum_{j=-r+1}^{\infty} |W_j|^2 h$ and $|W|^2 = \sum_{i=1}^{N} (W^{(i)})^2$.

Assumption 1.2. The matrices $\{A_{m\sigma}\}_{m=-r,\sigma=-1}^{p\ \ s}$ are simultaneously diagonalizable.

Definition 1.1 We say the finite difference model Q is initial and boundary value l_2-stable, if for any given time $T > 0$, there exist constants $M > 0$ and $\tau_0 > 0$ such that, for any initial value f^σ and boundary value $g_\mu (0 \le \sigma \le s, -r+1 \le \mu \le 0)$, the estimate

$$\|U^n\|_x^2 \le M\Big(\sum_{\mu=-r+1}^{0} \|g_\mu\|_{t \le T}^2 + \sum_{\sigma=0}^{s} \|f^\sigma\|_x^2 \Big)$$

holds for all n and τ with $n\tau \le T, \tau \le \tau_0$. Here $\|g_\mu\|_{t \le T}^2 = \sum_{n=s}^{T/\tau} |g_\mu|^2 k$.

The aim of this paper is to show that, for many commonly used difference models defined by (1.3), the necessary and sufficient condition for GKS-stability (in the sense of definition 3.3 of [1]) is a necessary condition for l_2-stability.

We shall use some results of [1] and [2], and assume that the reader is familiar with those papers.

§2. Left-going and Right-going Signals

Let Q denote the difference scheme

$$Q_{-1} U_j^{n+1} = \sum_{\sigma=0}^{s} Q_\sigma U_j^{n-\sigma},$$