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CONVERGENCE THEORY FOR AOR METHOD*

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Abstract

In this paper we give some sufficient conditions for the convergence of the AOR method, introduced by Hadjidimos [5], which include the ones from [1], [2], [5], [6], [7], [9], [10], [11] and [12] and which show that the necessary condition given in [8] for the convergence of the AOR method is not valid. We give general conditions for the class of H-matrices, but they are not always easy to check in practice. Consequently, we give some more practical conditions concerning some subclasses of H-matrices.

§1. Introduction

Among the various iterative methods which are used for the numerical solution of the linear system

$$Ax = b,$$

where $A \in C^{n,n}$ is a nonsingular matrix with nonzero diagonal entries, and $x, b \in C^n$ with x unknown and b known, the completely consistent linear stationary iterative schemes of first degree play a very important role. Such an iterative method, called the accelerated overrelaxation (AOR) method, was introduced by Hadjidimos in [5]. Since the introduction of the AOR method, many properties as well as unmerical results concerning this method have been given. There are many papers dealing with the linear systems with a matrix which is strictly diagonally dominant (SDD), irreducible diagonally dominant (IDD), or generalized diagonally dominant (GDD) is an M– or H– matrix (cf. [1], [5], [6], [9], [10], [11], [12], [17], [18]). in [2] and [7] some new classes of linear systems have been considered. The purpose of this paper is: i) to present some further basic results concerning the convergence of the AOR method when the matrix A is an H-matrix (all of the mentioned classes are H-matrices), and ii) to give more practical sufficient conditions for the convergence of the AOR method when the matrix A belongs to some special subclasses of H-matrices.

Let $A = D - T - S$ be the decomposition of the matrix A into its diagonal, strictly lower and strictly upper triangular parts, respectively and let $\omega, \sigma \in R, \omega \neq 0$. The associated AOR method can be written as

$$x^{k+1} = M_{\sigma,\omega} x^k + d, \quad k = 0, 1, \cdots, x^0 \in C^n,$$

where $M_{\sigma,\omega} = (D - \sigma T)^{-1}((1-\omega)D + (\omega - \sigma)T + \omega S), \quad d = \omega(D - \sigma T)^{-1} b.$

Some special cases of this method are

$$\text{AOR} \begin{cases} \xrightarrow{\omega = \sigma} \text{SOR} \xrightarrow{\omega = 1} \text{Gauss-Seidel} \\ \xrightarrow[\sigma = 0]{} \text{JOR} \xrightarrow[\omega = 1]{} \text{Jacobi} \end{cases}$$

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The AOR method has some connection with the extrapolation principle, since it is an extrapolation of either the Jacobi method (case $\sigma = 0$) or the SOR method (case $\sigma \neq 0$, where the extrapolation parameter is ω/σ). This fact and many numerical examples (cf. [1], [5]) show the superiority of the AOR method.

§2. Preliminaries

We shall use the following notations:

$$N = \{1, 2, \cdots, n\}, \quad N(i) = N \setminus \{i\}, \quad i \in N.$$

For any matrix $A = [a_{ij}] \in C^{n,n}$ (= set of all complex $n \times n$ matrices) and $i \in N, \alpha \in [0, 1]$, we define

$$P_i(A) = \sum_{j \in N(i)} |a_{ij}|, \quad Q_i(A) = \sum_{j \in N(i)} |a_{ji}|,$$

$$P_{i,\alpha}(A) = \alpha P_i(A) + (1 - \alpha) Q_i(A), \quad Q_i^*(A) = \max_{j \in N(i)} |a_{ji}|,$$

$$Q_i^{(r)}(A) = \max_{t_r \in \theta_r} \sum_{j \in t_r} |a_{ji}|,$$

where $r \in N$ and θ_r is the set of all choices $t_r = \{i_1, \cdots, i_r\}$ of different indices from N.

Definition 2.1. *A real square matrix whose off-diagonal elements are all non-positive is called L-matrix.*

Definition 2.2. *A regular L-matrix A for which $A^{-1} \geq 0$ is called M-matrix.*

In [3] we have proved the following two theorems.

Theorem 2.1. *Let A be an L-matrix, whose diagonal elements are all positive such that at least one of the following conditions is satisfied:*

(i) $a_{ii} > P_i(A), i \in N$(SDD).

(ii) $a_{ii} > P_{i,\alpha}(A), i \in N$, for some $\alpha \in [0, 1]$.

(iii) $a_{ii} > P_i^{\alpha}(A) Q_i^{1-\alpha}(A), i \in N$, for some $\alpha \in [0, 1]$.

(iv) $a_{ii} a_{jj} > P_i(A) p_j(A), i \in N, j \in N(i)$.

(v) $a_{ii} a_{jj} > P_i^{\alpha}(A) Q_i^{1-\alpha} p_j^{\alpha}(A) Q_j^{1-\alpha}(A), i \in N, j \in N(i)$, for some $\alpha \in [0, 1]$.

(vi) For each $i \in N$ it holds that $a_{ii} > P_i(A)$ or

$$a_{ii} + \sum_{j \in J} a_{jj} > Q_i(A) + \sum_{j \in J} Q_j(A), \text{ where } J := \{i \in N : a_{ii} \leq Q_i(A)\}.$$

(vii) $a_{ii} > \min(P_i(A), Q_i^*(A)), i \in N$ and $a_{ii} + a_{jj} > P_i(A), i \in N, j \in N(i)$.

(viii) $a_{ii} > Q_i^{(p)}(B), i \in N$ and $\sum_{j \in t_p} a_{ii} > \sum_{j \in t_p} P_i(A), t_p \in \theta_p$, for some $p \in N$.

(ix) There exists $i \in N$ such that

$$a_{ii}(a_{jj} - P_j(A) + |a_{ji}|) > P_i(A)|a_{ji}|, j \in N(i).$$

Then A in an M-matrix.

Note that SDD matrices satisfy all of the conditions (i)–(ix).

For any matrix $A = [a_{ij}] \in C^{n,n}$, we define $M(A) = [m_{ij}] \in R^{n,n}$ as follows

$$m_{ii} = |a_{ii}|, i \in N, m_{ij} = -|a_{ij}|, i \in N, j \in N(i).$$