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BLOCK IMPLICIT HYBRID ONE-STEP METHODS*

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Abstract

A class of k-block implicit hybrid methods for solving the initial value problem for ordinary differential equations are studied, which take a block of k new values at each step. These methods are examined for the property of A-stability. It is shown that the method of order $2k+2$ exists uniquely, and these methods are A-stable for block sizes $k = 1, 2, \ldots, 5$.

§1. Introduction

We shall study a class of methods for solving numerically the initial value problem for ordinary differential equations. These methods are named k-block implicit hybrid one-step methods, and take k new values at each step.

Block methods have been studied by a number of authors, such as Rosser, Shampine and Watts, Bichart and Picel, and Zhou Bing. Shampine and Watts[6], [7] did further research on theori s of block methods. They presented a different approach based on interpolatory formulas of Newton-Cotes type; the methods are of order $k+1$ for k odd and $k+2$ for k even. They also showed that the methods are A-stable for sizes $k = 1, 2, \ldots, 8$.

The fatal defect of block methods is inversion of a $km \times km$ matrix during Newton iterations, where m is the number of differential equations. So the use of higher order block methods is limited. To avoid the defect, we present a class of block implicit hybrid one-step methods, which are combinations of hybrid methods with block methods. These methods with small k possess higher accuracy and good stability. It is shown that the method of order $2k+2$ exists uniquely, and these methods are A-stable for block sizes $k = 1, 2, \ldots, 5$.

§2. A General Formulation and Convergence

Consider the initial value problem

$$y' = f(x, y), \quad y(\alpha) = \eta, \quad \alpha \le x \le \beta. \tag{2.1}$$

Let $x_{n+i} = x_n + ih, x_{n+v_i} = x_n + v_i h$, where $n = mk, m = 0, 1, 2, \ldots, i = 1, 2, \ldots, k$, and $v_i \notin Z, i = 1, 2, \ldots, k, v_1 < v_2 < \ldots < v_k$. Let y_j be the approximation of $y(x_j)$. Then the formulas are in the form

$$\begin{cases} Y_m = y_n K^0 + hBF(Y_m) + hf_n b + hDF(Y_{m+v}), \\ Y_{m+v} = -A_* Y_m - y_n a_* + hB_* F(Y_m) + hf_n b_*, \end{cases} \tag{2.2}$$

where $f_j = f(x_j, y_j), k^0 = (1, \ldots, 1)^T, B, D, A_*, B_* \in R^{k\times k}, b, a_*, b_* \in R^{k\times 1}$, D is nonsingular, $Y_m = (y_{n+1}, \ldots, y_{n+k})^T, Y_{m+v} = (y_{n+v_1}, \ldots, y_{n+v_k})^T, F(Y_m) = (f_{n+1}, \ldots, f_{n+k})^T$, $F(Y_{m+v}) = (f_{n+v_1}, \ldots, f_{n+v_k})^T$.

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Equation (2.2) is a system of nonlinear equations for Y_m and can be written as

$$Y_m = y_n K^0 + hBF(Y_m) + hf_n b + hDF(-A_* Y_m - y_n a_* + hB_* F(Y_m) + hf_n b_*) \equiv G(Y_m);$$

thus

$$G'(Y_m) = h[BF'(Y_m) + DF'(Y_{m+v})(-A_* + hB_* F'(Y_m))].$$

If h is suitably small, we have $\|G'(Y_m)\| < 1$. Then (2.2) has a unique solution. In practice, we may have to presume the existence of a solution.

With the method (2.2), we define two linear difference operator vectors $\mathcal{L}$ and $\mathcal{L}^*$ by

$$\mathcal{L}[Y_m(x); h] = Y_m(x) - y(x)K^0 - hBY'_m(x) - hy'(x)b - hDY'_{m+v}(x), \tag{2.3}$$

$$\mathcal{L}^*[Y_m(x); h] = Y_{m+v}(x) + A_* Y_m(x) + y(x)a_* - hB_* Y'_m(x) - hy'(x)b_*, \tag{2.4}$$

where $Y_m^{(i)}(x) = (y^{(i)}(x+h), \cdots, y^{(i)}(x+kh))^T, Y_{m+v}^{(i)}(x) = (y^{(i)}(x+v_1 h), \cdots, y^{(i)}(x+v_k h))^T, i = 0, 1$. Expanding $y(x+ih), y(x+v_i h)$ and their derivatives as Taylor series about x and collecting terms in (2.3) and (2.4) give

$$\mathcal{L}[Y_m(x); h] = y(x)c_0 + hy'(x)c_1 + \cdots + h^p y^{(p)}(x)c_p + \cdots, \tag{2.5}$$

$$\mathcal{L}^*[Y_m(x); h] = y(x)c_0^* + hy'(x)c_1^* + \ldots + h^q y^{(q)}(x)c_q^* + \cdots, \tag{2.6}$$

where c_p and c_q^* are constant vectors. Comparing (2.3) and (2.4) with (2.5) and (2.6), we have

$$\begin{cases} c_0 = 0, \\ c_1 = K - BK^0 - b - Dv^0, \\ c_p = K^p/p! - BK^{p-1}/(p-1)! - Dv^{p-1}/(p-1)!, \quad p = 2, 3, \ldots, \end{cases} \tag{2.7}$$

$$\begin{cases} c_0^* = v^0 + A_* K^0 + a_*, \\ c_1^* = v + A_* K - B_* K^0 - b_*, \\ c_q^* = v^q/q! + A_* K^q/q! - B_* K^{q-1}/(q-1)!, \quad q = 2, 3, \ldots, \end{cases} \tag{2.8}$$

where $K^s = (1^s, 2^s, \ldots, k^s)^T$ and $v^s = (v_1^s, v_2^s, \ldots, v_k^s)^T$. For formula (2.2), a convergence theorem can be easily obtained.

Theorem 1. *Suppose the method is defined by (2.2), and the linear difference operator vectors $\mathcal{L}$ and $\mathcal{L}^*$ satisfy $\|\mathcal{L}\| = O(h^{p+1})$ and $\|\mathcal{L}^*\| = O(h^{q+1})$. Then the method is convergent with global error of order h^r where $r = min(p, q+1)$, and the method is said to be of order r.*

In order to obtain a high order method, we choose $B, D, A_*, B_*, b, v, a_*, b_*$, as follows:

$$b = K - BK^0 - Dv^0, \tag{2.9a}$$

$$K^p/p! - BK^{p-1}/(p-1)! - Dv^{p-1}/(p-1)! = 0, \quad p = 2, 3, \ldots, 2k+2; \tag{2.9b}$$

$$\begin{cases} a_* = -v^0 - A_* K^0, \\ b_* = v + A_* K - B_* K^0, \end{cases} \tag{2.10a}$$

$$v^q/q! + A_* K^q/q! - B_* K^{q-1}/(q-1)! = 0, \quad q = 2, 3, \ldots, 2k+1. \tag{2.10b}$$