Journal of Computational Mathematics, Vol.9, No.4, 1991, 330–336.

THE SPECTRAL METHOD FOR THE GENERALIZED KURAMOTO-SIVASHINSKY EQUATION*

Guo Bo-ling
(*Institute of Applied Physics and Computational Mathematics, Beijing, China*)

Wu Xiang-hui
(*Zhongshan University, Guangzhou, China*)

Abstract

A spectral method is proposed, the existence and uniqueness of the global and smooth solution are proved for the periodic initial value problem of the generalized K-S equation. The error estimates are established and the convergence is proved for the approximate solution of the spectral method.

§1. Introduction

The Kuramoto-Sivashinsky equation

$$\Phi_t + \Phi_x^2 + \Phi_{xx} + \Phi_{xxxx} = 0 \tag{1.1}$$

was independently advocated by Kuramoto[1] in connection with reaction-diffusion systems, and then by Sivashinsky[2] in modeling flame propagation; it also arises in the context of viscous film flow[3], bifurcating solutions of the Navier-Stokes equation[4], etc.

Differentiating (1.1) with respect to x and setting $u = \Phi_x$, we get

$$u_t + (u^2)_x + u_{xx} + u_{xxxx} = 0. \tag{1.2}$$

In the present paper, we consider the generalized K-S equation of the form

$$u_t + f(u)_x + \alpha u_{xx} = \beta u_{xxxx} = g(u) \tag{1.3}$$

and its periodic initial value problem

$$u(x,0) = u_0(x), \quad x \in R^1, \quad u(x,t) = u(x+2\pi, t), \quad x \in R^1, \quad t \geq 0 \tag{1.4}$$

where $\alpha, \beta > 0$ are constants.

We propose a spectral method for the problem (1.3)–(1.4), prove the existence of the global smooth solution for the problem (1.3)–(1.4), and establish the error estimates and convergence for the approximate solution.

* Received August 4, 1988.

§2. The Spectral Methods and a Priori Estimates

Here we adopt the usual notation and convention. Let $\Omega = [0, 2\pi]$, $L_p(\Omega)$ denotes the Lebesgue space with the norm $\|u\|_{L_p} = \left(\int_0^{2\pi} |u|^p dx\right)^{1/p}$. If we define the inner product

$$(u, v) = \int_0^{2\pi} u(x)v(x)dx, \quad \|u\|_{L_2}^2 = (u, u),$$

then $L_2(\Omega)$ is a Hilbert space; especially, $L_\infty(\Omega)$ denotes the Lebesgue space with norm $\|u\|_{L_\infty} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. Let $H^m(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|_{L_2}^2\Big)^{1/2} \quad \text{or simply} \quad \|u\|_m.$$

Let $L^\infty(0, T; H^m)$ denote the space of the functions $u(x, t)$ each of which belongs to H^m as a function of x for every fixed $t, 0 \le t \le T$, and $\sup_{0\le t\le T} \|u(\cdot, t)\|_m < \infty$.

Let $H_p^m(\Omega) = \{u(x) | u \in H^m(\Omega),\ u^j(x) = u^j(x + 2\pi),\ 0 \le j \le m - 1\}$ be a periodic functional space, where $u^j = \dfrac{d^j u}{dx^j}$, $S_N = \text{Span}\ \{w_j(x), 1 \le j \le N\}$ is a subspace spanned on the basis $\{w_j(x)\}$, $j = 1, \cdots, N$, where $w_j(x) = \exp\{ijx\}$, $i = \sqrt{-1}$.

We construct an approximate solution of problem (1.3)–(1.4) as follows:

$$U_N(x, t) = \sum_{j=-N}^{N} \gamma_{jN}(t)w_j(x), \quad x \in \Omega$$

where the coefficient functions $\gamma_{jN}(t)$ should satisfy the equations

$$(U_{Nt} + f(U_N)_x + \alpha U_{Nxx} + \beta U_{Nxxxx}, w_j) = (g(U_N), w_j) \tag{2.1}$$

with the initial condition

$$U_N(x, 0) = U_{0N}(x), \quad x \in \Omega \tag{2.2}$$

where

$$U_{0N}(x) \xrightarrow{H^2(\Omega)} u_0(x) \quad \text{as} \quad N \to \infty.$$

Problem (2.1)–(2.2) can be considered as an initial value problem of nonlinear ordinary differential equations of first order with unknown functions $\gamma_{jN}(t)$. Under the conditions of the lemmas and the a priori estimates in the present section, we know that there exists a global solution in the interval $[0, T]$ for the initial value problem (2.1)–(2.2).

Now we make the a priori estimates for the solution of problem (2.1)–(2.2).

Lemma 1. *If the following conditions are satisfied* :

(i) $f(u) \in C^1$, $\alpha > 0$, $\beta > 0$, (ii) $g(0) = 0, g'_u \le b$, (iii) $u_0(x) \in L_2(\Omega)$,

then for the solution $U_N(x, t)$ of problem (2.1)–(2.2) there is the estimate

$$\|U_N\|_{L^\infty(0,T;L_2(\Omega))} + \|U_{Nxx}\|_{L^2(0,T;L_2(\Omega))} \le E_0 \tag{2.3}$$

where the constant E_0 is independent of N.

Proof. Multiplying (2.1) by $\gamma_{jN}(t)$ and summing them up for j from 1 to N, we have

$$(U_{Nt} + f(U_N)_x + \alpha U_{Nxx} + \beta U_{Nxxxx}, U_N) = (g(U_N), U_N).$$