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AN ISOMETRIC PLANE METHOD FOR LINEAR PROGRAMMING*1)

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Abstract

In this paper the following canonical form of a general LP problem,

$$\max Z = C^T X,$$
$$\text{subject to } \ AX \geq b$$

is considered for $X \in R^n$. The constraints form an arbitrary convex polyhedron Ω^m in R^n. In Ω^m a strictly interior point is successively moved to a higher isometric plane from a lower one along the gradient defined in the paper. Finally, the highest boundary point which makes the objective function value maximum is found or the infinite value of the objective function is concluded. For an $m * n$ matrix A the arithmetic operations of a movement are $O(mn)$ in our algorithm. The algorithm enables one to solve linear equations with ill conditions as well as a general LP problem. Some interesting examples illustrate the efficiency of the algorithm.

§1. Introduction

The following cononical form of a general LP problem

$$\max Z = C^T X, \quad \text{subject to } AX \geq b \tag{1.1}$$

is considered in this paper, where C^T denotes the transpose of $C, C = (c_1, c_2, \cdots, c_n)^T$, $X = (x_1, x_2, \cdots, x_n)^T$, $A = (a_{ij}), i = 1, \cdots, m, j = 1, 2, \cdots, n, b = (b_1, \cdots, b_m)^T$. Note that problem (1.1) is called general LP problem because $X \geq 0$ is not required specially in the constraints, that is, the constraints of problem (1.1) can form an arbitrary convex polyhedron in an n-dimensional space R^n. The convex polyhedron is an empty set if the constraints of (1.1) are inconsistent. In this paper, (1.1) is said to have no solution if and only if the constraints are inconsistent.

Although L.G.Khachiyan published the first polynomial-time algorithm for linear programming, the ellipsoid algorithm[3], in 1979, the methods which are practically efficient are the simplex method and the method presented by N.Karmarkar in 1984[1,2,5]. In 1947, G.B.Dantzig designed his famous simplex method, but the idea can be traced back to J.B.J.Fourier in the 1820s[1,5]. In showing how to find the best L_∞ approximation to a

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solution of linear equations, he used the notion of a point moving along the edges of a polyhedron from one vertex to another[1].

If in the simplex method the objective point is moved along a broken line on the boundary of the convex polyhedron, then Karmarkar's algorithm is a method in which the objective point is moved along a curve in the convex polyhedron. The isometric plane method presented in this paper makes the objective point move from one isometric plane to another along a broken line in the convex polyhedron, and the objective function maximizes. In our view, the simplex method is a special case of the isometric plane method.

Compared with existing methods, the isometric plane method solves problem (1.1) while not increasing dimensionality n of the space and number m of the constraints. It needs no slack variables or duality principle.

Throughout the paper we always suppose that the dimension of the vector space R^n, $n \geq 2$, the number of constraints $m \geq 1$, and

$$C_i^T = (a_{i1}, a_{i2}, \cdots, a_{in}) \neq 0 \ , \ i = 1, \cdots, m \ ,$$

and naturally $C \neq 0$.

§2. Description of the Method

The m constraints of problem (1.1) form a convex polyhedron Ω^m in R^n. Set

$$C_i^T = (a_{i1}, a_{i2}, \cdots, a_{in}) \ , \quad i = 1, \cdots, m \ . \tag{2.1}$$

The polyhedron Ω^m may be denoted as

$$\Omega^m = \{X \in R^n \mid C_i^T X \geq b_i \ , \quad i = 1, \cdots, m\} \ . \tag{2.2}$$

We first suppose that Ω^m is nonempty and has interior points.

The boundary $\partial\Omega^m$ of Ω^m consists of all or part of hyperplanes

$$P_i = \left\{X \in R^n \middle| \sum_{j=1}^{n} a_{ij}x_j = b_i \ , \ 1 \leq i \leq m\right\}. \tag{2.3}$$

The normal vector of P_i is C_i denoted by (2.1). The positive direction of vector C_i passing through X^i directs to the inside of Ω^m. The intersection of P_i and $\partial\Omega^m$ is called a surface of Ω^m, denoted as

$$\bar{P}_i = P_i \cap \partial\Omega^m \ .$$

Regarding C in the objective function of (1.1) as a vector, we define the c-line passing through Y :

$$L_y^c = \{X \in R^n \mid X - Y = tC, \ t \in R^1\} \ , \tag{2.4}$$

and denote $t > 0$ as the positive direction of L_y^c. The objective function value in (1.1) is increased with a point moving forward in the positive direction of L_y^c from Y. Therefore,