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COMPUTE MULTIPLY NONLINEAR EIGENVALUES*

Li Ren-cang
(*Institute of Applied Physics and Computational Mathematics, Beijing, China*)

Abstract

An incomplete QR decomposition called QR-like decomposition is proposed and studied. The developed theory enables us to construct two new algorithms for computing multiply nonlinear eigenvalues.

Several numerical tests are presented to illustrate their behavior in comparison with Kublanovskaya's approach.

§1. Introduction

Consider an n by n matrix $A(\lambda)$ whose entries $a_{ij}(\lambda)$ are analytic functions of a complex scalar λ (cf. $a_{ij}(\lambda)$ are functions which have at least first order derivatives of real scalar λ). We shall call such a matrix a functional λ-matrix[5]. However, if $a_{ij}(\lambda)$ are polynomials in λ, then $A(\lambda)$ is commonly known as a λ-matrix[8]. Values λ and the corresponding nonzero vectors x and y which satisfy

$$A(\lambda)x = 0, \quad y^H A(\lambda) = 0 \tag{1.1}$$

are the solutions to the nonlinear eigenvalue problem associated with $A(\lambda)$. In the above equations λ is known as a nonlinear eigenvalue, x a right and y a left nonlinear eigenvector (for the meaning of superscript H, see Notation below). Obviously, all the nonlinear eigenvalues of $A(\lambda)$ are also the roots of its characteristic equation

$$\det A(\lambda) = 0, \tag{1.2}$$

and vice versa.

By now, several iterative methods (see [5]-[9] and [12]) have been proposed for solving equations (1.1). Roughly speaking, they are just effective for computing simple nonlinear eigenvalues, and often converge slowly when a multiply nonlinear eigenvalue arises. In §2 we shall propose a kind of incomplete QR decomposition called QR-like decomposition, and then extend the differentiability theory for QR

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decomposition (QRD) (see [6], [9] and [11]) to such decomposition. The extended theory is the basis of our algorithms for computing multiply nonlinear eigenvalue of $A(\lambda)$ in §3. In §4 and 5 we shall analyse the convergence theory of our algorithms. Numerical tests were made and are listed in §6 to illustrate their behavior.

Notation. We shall use $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) for the m by n complex (real) matrix set, $\mathbb{C}^m = \mathbb{C}^{m\times 1}$ ($\mathbb{R}^m = \mathbb{R}^{m\times 1}$), $\mathbb{C} = \mathbb{C}^1$ ($\mathbb{R} = \mathbb{R}^1$); $\mathcal{U}_n \subset \mathbb{C}^{n\times n}$ denotes the n by n unitary matrix set. $I^{(n)}$ is the n by n unit matrix, $e_j^{(n)}$ the jth column of $I^{(n)}$ and $I_j^{(n)} \equiv (e_1^{(n)}, \cdots, e_j^{(n)})$, $K_j^{(n)} \equiv (e_{n-j+1}^{(n)}, \cdots, e_n^{(n)})$. When no confusion arises, these superscripts (n) are usually omitted. A^H, A^T denote the conjugate transpose and transpose of A respectively, and $\|A\|_F$, $\|A\|_2$ the Frobenius norm and the spectral norm of A, respectively. For a matrix $A = (a_1, \cdots, a_n)$ where a_i are n column vectors, we define a column vector col A by col $A \equiv (a_1^T, \cdots, a_n^T)^T$. An adhoc notation is that for a given integer t and a matrix $A \in \mathbb{K}^{n\times n}$ we always partition it as

$$A = \begin{array}{c} \\ n-t \\ t \end{array}\!\!\begin{array}{c} \begin{array}{cc} n-t & t \end{array} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array}, \tag{1.3}$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Symbol $\otimes$ denotes the Kronecker product of matrices.

§2. QR-Like Decompositions

In this section, we first propose a kind of incomplete QR decompositions (QRDs), and then study its differentiability properties, which are the bases of our algorithms for computing multiply nonlinear eigenvalues in the forthcoming sections.

Definition 2.1. *Let $B \in \mathbb{C}^{n\times n}$ and $1 \le t \le n$ be an integer. A decomposition*

$$B = QR, \quad Q \in \mathcal{U}_n \tag{2.1}$$

is called a QR-*like decomposition with the index t* (QRD(t)), *if*

$$R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}, \tag{2.22}$$

where $R_{11} \in \mathbb{C}^{(n-t)\times(n-t)}$ is an upper triangular matrix.

From this definition, it follows that a QRD is always a QRD (t), but generally the converse is not true if $t \neq 1$, and if $t = 1$, QRD and QRD (1) are equivalent.

The following theorem is an extension of Theorem 2.1 in [9].