Journal of Computational Mathematics, Vol.12, No.2, 1994, 132–137.

ON AN ESSENTIAL ESTIMATE IN THE ANALYSIS OF DOMAIN DECOMPOSITION METHODS*1)

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Abstract

A class of nonconforming finite elements is considered in this paper, which is continuous only at the nodes of the quasi-uniform mesh. We show that there exists an essential estimate which indicates the equivalence relation, independent of the mesh parameter, between the energies of the nonconforming discrete harmonic extensions in different subdomains. The essential estimate is of great importance in the analysis of the nonoverlapping domain decomposition methods applied to second order partial differential equations discretized by nonconforming finite elements.

1. Main Result

Let Ω be a bounded connected open domain in R^2 with a piecewise smooth boundary $\partial\Omega$, $a_{ij}(x), i,j=1,2$, piecewise smooth bounded functions in Ω, and $(a_{ij}(x))$ a symmetric, uniformly positive definite matrix in Ω. Ω is divided into two open subdomains Ω_1, Ω_2 by an open smooth curve Γ , which satisfies

$$\Omega_1 \cap \Omega_2 = \phi, \quad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}, \quad \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma.$$

We make the following definitions:

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \tag{1.1}$$

$$a_k(u,v) = \int_{\Omega_k} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \quad k=1,2,$$

$$V_H = \{(v_1,v_2) : v_k \in H^1(\Omega_k), \quad v_k|_{\partial\Omega_k \cap \partial\Omega} = 0, \quad v_1|_\Gamma = v_2|_\Gamma,$$

$$a_k(v_k,\theta) = 0, \quad \forall \theta \in H_0^1(\Omega_k), \quad k=1,2\}.$$

v_k is called the a_k harmonic extension in $\Omega_k, k=1,2$, if $(v_1,v_2) \in V_H$. Using the trace theorem and a well-known priori inequality[1,6], we obtain

* Received April 8, 1993.
1) The Project Supported by National Natural Science Foundation of China.

Theorem 1[1]. *There exist two positive constants σ_1, τ_1, such that*

$$\tau_1 a_2(v_2, v_2) \le a_1(v_1, v_1) \le \sigma_1 a_2(v_2, v_2), \quad \forall (v_1, v_2) \in V_H.$$

In what follows, for simplicity, we assume that Ω is a bounded polygonal domain in R^2 with a quasi-uniform mesh $\Omega_h = \{e\}$, where e , a triangle or a quadrilateral, represents the typical element in Ω_h . Let Ω_h be compatible with the subdomain division, i.e.

$$e \cap \Gamma = \phi, \quad \forall e \in \Omega_h$$

Let $S^h(\Omega)$ be a conforming finite element space, e.g. the space of continuous piecewise linear or bilinear functions defined relative to the mesh Ω_h. We define

$$\begin{aligned}
&S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega), \quad \Gamma_h = \Omega_h|_\Gamma, \\
&S^h(\Omega_k) = S_0^h(\Omega)|_{\overline{\Omega}_k}, \quad k = 1, 2, \\
&S_0^h(\Omega_k) = S^h(\Omega_k) \cap H_0^1(\Omega_k), \quad k = 1, 2, \\
&V_C = \{(v_1^h, v_2^h) : v_k^h \in S^h(\Omega_k), \quad v_1^h|_{\Gamma_h} = v_2^h|_{\Gamma_h}, \\
&\qquad a_k(v_k^h, \theta_h) = 0, \quad \forall \theta_h \in S_0^h(\Omega_k), \quad k = 1, 2\}.
\end{aligned}$$

v_k^h is called the conforming discrete a_k harmonic extension in Ω_k, $k = 1, 2$, if $(v_1^h, v_2^h) \in V_C$. Concerning V_C, we have

Theorem 2[2]. *If the mesh Ω_h is quasi-uniform, then there exist two positive constants σ_2, τ_2 , independent of the mesh parameter h, such that*

$$\tau_2 a_2(v_2^h, v_2^h) \le a_1(v_1^h, v_1^h) \le \sigma_2 a_2(v_2^h, v_2^h), \quad \forall (v_1^h, v_2^h) \in V_C.$$

Theorems 1 and 2 are the foundations of the analysis of nonoverlapping domain decomposition methods applied to second order partial differential equations in the continuous case and in the conforming discrete case respectively. The aim of this paper is to show that an estimate , similar to *Theorem* 1 and *Theorem* 2, is ture for a class of nonconforming finite elements.

Let $T^h(\Omega)$ be the set $\{v^h : v^h = w^h + u^h, w^h \in \hat{T}^h(\Omega),\ u^h|_e$ is a finite order polynomial, $\forall e \in \Omega_h, u^h(x) = 0, \forall$ node $x \in \Omega_h\}$, where $\hat{T}^h(\Omega) = \{v^h \in C(\Omega) : v^h|_e$ is a linear function if e is a triangle, or a bilinear function if e is a quadrilateral, $\forall e \in \Omega_h\}$. Here, a node $x \in \Omega_h$ is defined to be the vertex of some $e \in \Omega_h$.

$T^h(\Omega)$ is a class of nonconforming finite elements, which is continuous only at the nodes of the mesh Ω_h, and many practical nonconforming elements possess this property (e.g. [3], [4], [5]). In what follows, we assume that the nonconforming approximate solution of a given problem exists uniquely and converges to the exact solution of the problem.