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AN ITERATIVE PROCEDURE FOR DOMAIN DECOMPOSITION METHOD OF SECOND ORDER ELLIPTIC PROBLEM WITH MIXED BOUNDARY CONDITIONS *1)

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Abstract

This paper is devoted to study of an iterative procedure for domain decomposition method of second order elliptic problem with mixed boundary conditions (i.e., Dirichlet condition on a part of boundary and Neumann condition on the another part of boundary). For the pure Dirichlet problem, Marini and Quarteroni [3], [4] considered a similar approach, which is extended to more complex problem in this paper.

1. Introduction

There has been a considerable number of recent developments in non-overlap domain decomposition techniques for second order elliptic problems. We refer especially to Marini and Quarteroni [3], [4] and the references therein. One of motivations for increasing interest in domain decomposition approach is to deal with different type of equations in different parts of the physical domain, such as in the mathematical modeling of elastic composite structures.

In this paper we study an iterative procedure for domain decomposition method of a simple second order elliptic problem with mixed boundary conditions, i.e., Dirichlet condition on a part of boundary and Neumann condition on the another part of boundary. For the pure Dirichlet problems, Marini and Quarteroni [3], [4] considered a similar approach, which is so called the D-N (Dirichlet-Neumann) alternative iteration, while our iterative scheme is so called the N-D (Neumann-Dirichlet) alternative iteration, which is appropriate to the mixed boundary value problems.

The outline of the paper is as follows. In Section 2, we introduce multidomain formulation and iterative scheme for a simple second order elliptic problem with mixed boundary conditions. In section 3, we present an harmonic extension lemma which is important for the analysis of convergence of the iterative scheme. Finally in Section 4, the convergence of the iterative scheme is proved, with different manner from one in [3], [4] in order to appropriate the mixed boundary value problem.

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2. Domain Decomposition Method for Second Order Elliptic Problem with Mixed Boundary Conditions

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$. Consider the following boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \; u = 0 \quad \text{on } \Gamma_0, \; \partial_\nu u = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0, \tag{2.1}$$

(c.f. Fig. 1), where $f \in L^2(\Omega)$, ν denotes the outward normal unit vector to $\partial\Omega$, $\partial_\nu u$ denotes the outward normal derivative.

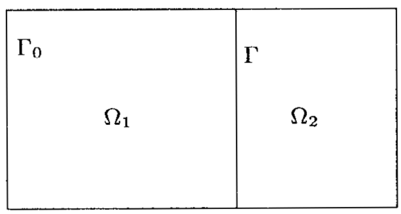

Fig.1

We assume that Ω is partitioned into two non-overlap subdomains Ω_1 and Ω_2, i.e., $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$, and we denote by Γ the common boundary of Ω_1 and Ω_2. It can be easily shown that the problem (2.1) is equivalent to the following split problems:

$$-\Delta u_1 = f \text{ in } \Omega_1, \; u_1 = 0 \text{ on } \Gamma_0, \; \partial_{\nu^1} u_1 = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0), \; \partial_{\nu^1} u_1 = -\partial_{\nu^2} u_2 \text{ on } \Gamma, \tag{2.2}$$

and

$$-\Delta u_2 = f \quad \text{in } \Omega_2, \; \partial_{\nu^2} u_2 = 0 \quad \partial\Omega_2 \setminus \Gamma, \; u_2 = u_1 \quad \text{on} \Gamma, \tag{2.3}$$

where $u_k = u|_{\Omega_k}$ for $k = 1, 2$, ν^k is the outward normal unit vector to $\partial\Omega_k$ (note that $\nu^1 = -\nu^2$ on Γ), and $\partial_{\nu^k} u_k$ $(k = 1, 2)$ is the outward normal derivative.

We now introduce an iterative procedure which is similar to that in [3], [4]. Let (c.f.[1],[2])

$$H^{\frac{1}{2}}(\Gamma) = \{\mu : \|\mu\|_{\frac{1}{2},\Gamma} < \infty\} \tag{2.4}$$

where

$$\|\mu\|_{\frac{1}{2},\Gamma} = \Big\{\|\mu\|_{0,\Gamma}^2 + \int_\Gamma \int_\Gamma \Big(\frac{\mu(s_x) - \mu(s_y)}{|s_x - s_y|}\Big)^2 ds_x ds_y\Big\}^{\frac{1}{2}}, \tag{2.5}$$

and

$$H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))' \text{–the duarity of the space } H^{\frac{1}{2}}(\Gamma). \tag{2.6}$$

Let $g^0 \in H^{-\frac{1}{2}}(\Gamma)$ be given. For $n \geq 1$ the sequence of functions u_1^n, u_2^n are constructed by iterative scheme with solving the following problems:

$$-\Delta u_1^n = f \text{ in } \Omega_1, \; u_1^n = 0 \text{ on} \Gamma_0, \; \partial_{\nu^1} u_1^n = 0 \text{ on } \partial\Omega_1 \setminus (\Gamma \cup \Gamma_0), \; \partial_\nu u_1^n = g^{n-1} \text{ on } \Gamma, \tag{2.7}$$