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ERROR ESTIMATES UNDER MINIMAL REGULARITY FOR SINGLE STEP FINITE ELEMENT APPROXIMATIONS OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This paper studies error estimations for a fully discrete, single step finite element scheme for linear parabolic partial differential equations. Convergence in the norm of the solution space is shown and various error estimates in this norm are derived. In contrast to like results in the extant literature, the error estimates are derived in a stronger norm and under minimal regularity assumptions.

Key Words. fully discrete approximation, parabolic equations, error estimate, finite element methods, backward Euler method.

1. Introduction

This paper is devoted to the study of error estimations for a fully-discrete, single step finite element approximation of linear parabolic equations of the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\,[A(\mathbf{x})\nabla u] = f(t,\mathbf{x}) \qquad \text{in } (0,T)\times\Omega \tag{1.1}$$

with the boundary and initial conditions

$$u = 0 \quad \text{on } (0,T)\times\partial\Omega, \qquad u(0,\mathbf{x}) = u_0(\mathbf{x}) \quad \text{in } \Omega\,, \tag{1.2}$$

where f is a given function and A is a matrix-valued, uniformly positive definite function. The fully discrete approximation scheme studied in this paper is a simple modification of the standard backward Euler method and it involves a temporal integral of the forcing term. This fully discrete scheme is well defined under minimal regularity assumptions on the forcing term and the initial condition; in particular, the forcing term f can be nondifferentiable in time, e.g., a temporal step function of the form $f = \sum_{i=1}^{J} f_i(t)\chi_{(t_i,t_{i+1})}(t)\Theta_i(\mathbf{x})$ where each (t_i,t_{i+1}) is a time interval in $[0,T]$ and $\chi_{(t_i,t_{i+1})}$ is the characteristic function for the interval (t_i,t_{i+1}) (such a choice of f corresponds to a setting in which different force patterns are applied on different time intervals.) The achievements of this paper include:

- fractional order error estimates in the norm of the solution space (this solution space will be made precise in Section 2.1) are derived under fractional order, uni-directional regularity assumptions on the forcing term;
- a first order δ error estimate (again in the norm of the solution space) is derived under standard assumptions that ensure solution regularity;
- convergence under minimal regularity.

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Compared to the convergence results and error estimates in the extant literature for linear parabolic PDEs (see, e.g., [2, 3, 4, 5, 9, 10, 17, 19, 20]), the convergence and error estimates in this paper are derived in a stronger norm and under weaker regularity assumptions.

The fractional order error estimates established in this paper are new and they allow us to prove the convergence under weaker regularity hypotheses on the forcing term and the initial condition than those assumed in standard convergence results in the literature. Other types of fractional order error estimates can be found in the literature, e.g., [14]. The fractional order error estimates of [14] were measured in the $H^{p,p/2}_{\alpha}((0,T)\times\Omega)$ norm (see [14] for this notation) with the forcing term belonging to $H^{2p-1,p-1/2}_{-1}((0,T)\times\Omega)$, whereas those of this paper are measured in the $H^{1,-1}((0,T)\times\Omega)$ norm and requires only uni-directional regularity.

The results of this paper can be used in conjunction with the Brezzi, Rappaz, Raviart theory (see, e.g., [6, 12]) to study fully discrete approximations of semilinear parabolic PDEs. The set-up of the fully-discrete approximations for semilinear PDEs is more involved and is illustrated in [13]. The fractional order error estimates under uni-directional regularity assumptions play a crucial role in the derivation of fully discrete error estimates for semilinear parabolic PDEs; see [13].

Another significant application of the results of this paper is to prove the convergence of and error estimates for fully discrete approximations of optimal control problems constrained by parabolic PDEs; such applications will be discussed elsewhere.

The rest of the paper is organized as follows. In §2, we introduce continuous and discrete (finite element) function spaces and define a weak formulation for the problem (1.1)–(1.2). In §3, we define semidiscrete and fully discrete finite element approximations to that problem. In §4, we derive estimates for the difference between the semidiscrete and fully discrete approximate solutions and, in §5, we establish, under minimal regularity assumptions, convergence of and error estimates for the fully discrete approximation.

2. Function spaces, finite element spaces, and weak formulations

2.1. Function spaces. We use the standard notations (see, e.g. [1]) for Sobolev spaces $W^{s,p}(\Omega)$ for all real s and $p\ge 1$, with their norms denoted by $\|\cdot\|_{W^{s,p}(\Omega)}$. When $p=2$, we use the notation $H^s(\Omega)=W^{s,2}(\Omega)$ for all real s, with their norm simply denoted by $\|\cdot\|_s$. We let $H^1_0(\Omega)$ stand for the completion of $C^\infty_0(\Omega)$ with respect to the $H^1(\Omega)$ norm. Note that $H^0(\Omega)=L^2(\Omega)$ so that the $L^2(\Omega)$ norm is denoted by $\|\cdot\|_0$. The inner products on $L^2(\Omega)$ is denoted by $[\cdot,\cdot]$, i.e.,

$$[u,v]=\int_\Omega uv\,d\mathbf{x}\qquad\forall\,u,v\in L^2(\Omega)\,.$$

The duality paring between a Banach space B and its dual will be generically denoted by $\langle\cdot,\cdot\rangle$.

For a $p\in[1,\infty]$, an interval $(a,b)\subset\mathbb{R}$, and a Banach space B with norm $\|\cdot\|_B$, we denote by $L^p(a,b;B)$ the set of measurable functions $v:(a,b)\to B$ such that $\int_a^b\|v(t)\|_B^p\,dt<\infty$. The norm on $L^p(a,b;B)$ for $p\in[1,\infty)$ is defined by

$$\|v\|_{L^p(a,b;B)}=\Big(\int_a^b\|v(t)\|_B^p\,dt\Big)^{\frac{1}{p}}\qquad\forall\,v\in L^p(a,b;B)\,.$$