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CONSTRAINED QUADRILATERAL NONCONFORMING ROTATED $\mathcal{Q}_1$ ELEMENT *1)

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Abstract

In this paper, we define a new nonconforming quadrilateral finite element based on the nonconforming rotated $\mathcal{Q}_1$ element by enforcing a constraint on each element, which has only three degrees of freedom. We investigate the consistency, approximation, superclose property, discrete Green's function and superconvergence of this element. Moreover, we propose a new postprocessing technique and apply it to this element. It is proved that the postprocessed discrete solution is superconvergent under a mild assumption on the mesh.

Mathematics subject classification: 65N30.
Key words: Constrained, Nonconforming Rotated $\mathcal{Q}_1$ element, Superconvergence, Postprocess.

1. Introduction

There are some lower order quadrilateral finite elements, e.g., the conforming isoparametric Q_1 element, the nonconforming rotated $\mathcal{Q}_1$ element and the nonconforming Wilson element. All these finite elements need at least four degrees of freedom. Recently, Park and Sheen have proposed a nonconforming quadrilateral P_1 element, which has only three degrees of freedom [10]. One of the key ideas of the P_1 element is that a linear function on a quadrilateral satisfies a constraint that the summation of values at the midpoints of one pair of opposite edges equals to the summation of values at the midpoints of the other pair of opposite edges.

In this paper, we define a new nonconforming quadrilateral finite element based on the nonconforming rotated $\mathcal{Q}_1$ element (NR $\mathcal{Q}_1$ hereafter)[9] by imposing a similar constraint on each element, the resulting element has only three degrees of freedom, too. We call this element constrained nonconforming rotated $\mathcal{Q}_1$ element(CNR $\mathcal{Q}_1$ for short). The CNR $\mathcal{Q}_1$ element and the P_1 element are equivalent on a rectangle, however, they are different on a general quadrilateral. We investigate some properties of this new element. A new postprocess technique is proposed to obtain a superconvergent discrete postprocessed solution.

The outline of the paper is as follows. In Section 2 and Section 3, we define the CNR $\mathcal{Q}_1$ element and apply it to the second order elliptic problem. In section 4, we define regular derivative Green function of nonconforming finite elements and investigate its properties. Section 5 is devoted to the analysis of the supperclose property and superconvergence of the CNR $\mathcal{Q}_1$ element. In Section 6 , we discuss the postprocessing technique which admits a superconvergent discrete postprocessed solution. This paper ends with numerical examples in Section 7.

We end this section with some notations. Let Ω be a convex polygon with the boundary $\partial\Omega$. We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ for $s \geq 0$ [1], the associated inner product is denoted by $(\cdot,\cdot)_s$, and the norm by $\|\cdot\|_s$ with the seminorm $|\cdot|_s$. $H^0(\Omega) = L^2(\Omega)$, in this case, the norm and inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$

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respectively. As usual, $H_0^s(\Omega)$ is the subspace of $H^s(\Omega)$ with vanishing trace on $\partial\Omega$. Define $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ equipped with the norm $\|\cdot\|_{-1}$, and $<\cdot,\cdot>$ denotes the dual pair between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. We shall also use the Sobolev spaces $W^{s,p}$ for $s \geq 0$ and $p \geq 1$, equipped with the norm $\|\cdot\|_{s,p,\Omega}$ with the seminorm $|\cdot|_{s,p,\Omega}$. If $p = 2$ we have $W^{s,p} = H^s(\Omega)$.

We use the standard gradient operator:

$$\nabla r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \quad \widehat{\nabla} r = \begin{pmatrix} \partial r/\partial \xi \\ \partial r/\partial \eta \end{pmatrix}.$$

Throughout this paper, C denotes a generic constant, which is not necessarily the same at different places, but independent of the mesh size h.

2. Constrained Nonconforming Rotated $\mathcal{Q}_1$ Element

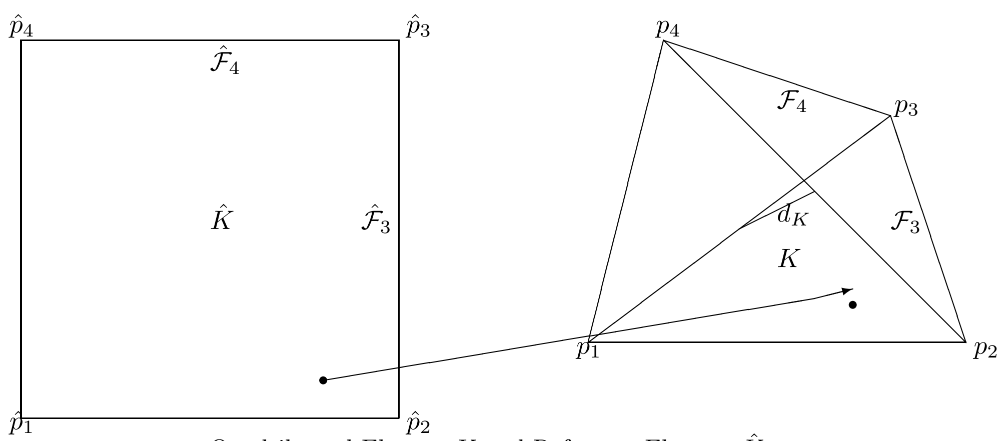
In this section, we introduce some notations and define a new nonconforming finite element method, namely, CNR $\mathcal{Q}_1$ element.

2.1 Quadrilateral Mesh

Let $J^h = \{K_i, i = 1, \cdots, Ne\}$ be a quasi-uniform quadrilateral partition of Ω with $\mathrm{diam}(K_i) \leq h$. Let N^V and N^E denote the numbers of nodes and elements of the partition, respectively, N_i^V and N_B^S denote the numbers of interior nodes and boundary edges, respectively.

We shall frequently use the following assumption on the partition J^h.

Assumption 2.1. *The distance d_K between the midpoints of two diagonals is of order $\mathcal{O}(h^{1+\alpha})$ with $1 \geq \alpha > 0$ when h tends to zero. If $\alpha = 1$, we obtain the usual Bi-section condition [11].*



Quadrilateral Element K and Reference Element $\hat{K}$

For a given element $K \in J^h$, its four nodes are denoted by $p_i(x_i, y_i), i = 1, \cdots, 4$ in the counterclockwise order. Let $\hat{K} = [-1, 1]^2$ denote the reference element with nodes $\hat{p}_i(\xi_i, \eta_i), i = 1, \cdots, 4$. Define the bilinear transformation $\mathcal{F}_K : \hat{K} \to K$ by

$$x = \sum_{i=1}^{4} x_i N_i(\xi, \eta), \quad y = \sum_{i=1}^{4} y_i N_i(\xi, \eta), \quad (\xi, \eta) \in \hat{K},$$