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ACCELERATION METHODS OF NONLINEAR ITERATION FOR NONLINEAR PARABOLIC EQUATIONS *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

This paper discusses the accelerating iterative methods for solving the implicit scheme of nonlinear parabolic equations. Two new nonlinear iterative methods named by the implicit-explicit quasi-Newton (IEQN) method and the derivative free implicit-explicit quasi-Newton (DFIEQN) method are introduced, in which the resulting linear equations from the linearization can preserve the parabolic characteristics of the original partial differential equations. It is proved that the iterative sequence of the iteration method can converge to the solution of the implicit scheme quadratically. Moreover, compared with the Jacobian Free Newton-Krylov (JFNK) method, the DFIEQN method has some advantages, e.g., its implementation is easy, and it gives a linear algebraic system with an explicit coefficient matrix, so that the linear (inner) iteration is not restricted to the Krylov method. Computational results by the IEQN, DFIEQN, JFNK and Picard iteration methods are presented in confirmation of the theory and comparison of the performance of these methods.

Mathematics subject classification: 65M06, 65M12.
Key words: Nonlinear parabolic equations, Difference scheme, Newton iterative methods.

1. Introduction

For solving the implicit scheme of nonlinear parabolic problems from various applications, iterative methods are used which adopt the inner-outer iteration mode. The inner iteration is the linear iterative methods for the linearized systems, and the outer cycle is the nonlinear iterative methods which will be discussed here. To a great extent the outer nonlinear iteration determines the accuracy and efficiency of the total solution procedure. In the energy conservative equation of the radiation hydrodynamics, the diffusion coefficients and the source term are nonlinear with respect to the temperature (the temperatures of radiation, ion or electron). During the construction of the linearization procedure, the key point is to preserve the characteristics of the original nonlinear parabolic equations so as to achieve high efficient solution. In [1]-[5], it is pointed out that the nonlinear convergence is tightly relevant to the selection of time step and the precision of solution. The efficient nonlinear iteration within one time step can speed up the convergence of the iteration solution greatly. So it is essential to find high efficient iterative methods in solving the nonlinear parabolic problems.

There are at least three reasons to prevent Newton methods applied in the nonlinear parabolic problems from some large scale scientific computations. The first is that the nonlinear

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iteration methods with super-linear convergent order often have local convergent region. In this regard, a common approach is to reduce the time step to ensure the nonlinear iteration method convergent. Actually, Newton method is sensitive to the iterative initial value, and can be regarded as a measure of the nonlinearity. However, reducing frequently the time step will increase a lot of computational time. The second reason is that the Newton method may change the features of the original partial differential equations, which makes the iteration hard to be convergent. A iteration method of preserving the characteristics of the original PDEs during the iteration process is more valuable than the one that possesses such property only at the end of the iteration procedure. The iteration method of keeping the parabolic feature of the nonlinear parabolic equations not only ensures the efficiency of the computation, but also keeps the iteration solution to be positive (see [7] for detail). Keeping the positivity in the iterative procedure is the foundation of the correct simulation of the physical problem. The third reason is that the Newton iteration should form a Jacobian matrix, which is often time-consuming, and is even impossible for some applications. For this issue, some papers (e.g. [4]-[6]) suggest applying the JFNK (Jacobian Free Newton-Krylov) method to deal with such problems.

In this paper, we pay attention to the last two reasons due to their importance. The main objective of this paper is that two new nonlinear iteration methods, called as the implicit-explicit quasi-Newton (IEQN) method and the derivative free implicit-explicit quasi-Newton (DFIEQN) method, are proposed. In these methods we construct a iterative (linearized) difference scheme from the nonlinear implicit scheme, instead of simply applying the Newton method or JFNK method to the nonlinear algebraic system of equations. In other words, the device of IEQN and DFIEQN methods are based on the nonlinear implicit scheme for the nonlinear parabolic equations, and not on the corresponding nonlinear algebraic system of equations. Moreover the performance of the DFIEQN method is examined along with some existing iteration methods including the semi-implicit method (SI), the fully implicit Picard method (FIP), fully implicit partial Newton method (FIPN) and the JFNK (Jacobian Free Newton-Krylov) method. Like JFNK method, the DFIEQN method is derivative free. But, unlike JFNK method our DFIEQN method has the advantage of FIP, i.e., its implementation is simple, and it gives a linear algebraic system with an explicit coefficient matrix, so that the inner iteration is not restricted to be chosen as the Krylov method and it is more convenient and efficient to get a preconditioner. Moreover we will prove the DFIEQN method is convergent quadratically, while the SI, FIP and FIPN is convergent linearly (see [7]).

The paper is organized as follows. Some nonlinear iterative methods are constructed in following section 2. These include the known semi-implicit (SI) method, the fully implicit Picard (FIP) method, and the fully implicit partial Newton (FIPN) method. And then we describe the construction of the implicit-explicit quasi-Newton (IEQN) method and the derivative free implicit-explicit quasi-Newton (DFIEQN) method. In the section 3 some assumptions and auxiliary lemmas are introduced, and the main convergence theorems are stated. In the section 4, we study the convergence property of the constructed nonlinear iteration method, in particular we will prove the 2nd order convergence of the IEQN and DFIEQN methods. In the last section, numerical results are presented to show the performance of these methods.

2. Construction of the Iteration Sequences

2.1. The Problem and Some Notations

To present the idea of the construction of the nonlinear iteration, the following one dimensional nonlinear parabolic problem is considered for simplicity here

$$u_t - (A(x,t,u)u_x)_x = f(x,t,u), \;\; Q_T = \{0 < x < l, 0 < t \leq T\} \tag{2.1}$$

$$u(x,0) = u^0(x), \;\; 0 \leq x \leq l \tag{2.2}$$

$$u(0,t) = u(l,t) = 0, \;\; 0 \leq t \leq T \tag{2.3}$$