Journal of Computational Mathematics, Vol.24, No.2, 2006, 193–208.

NATURAL BOUNDARY ELEMENT METHOD FOR THREE DIMENSIONAL EXTERIOR HARMONIC PROBLEM WITH AN INNER PROLATE SPHEROID BOUNDARY *1)

Hong-ying Huang and De-hao Yu
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, we study natural boundary reduction for Laplace equation with Dirichlet or Neumann boundary condition in a three-dimensional unbounded domain, which is the outside domain of a prolate spheroid. We express the Poisson integral formula and natural integral operator in a series form explicitly. Thus the original problem is reduced to a boundary integral equation on a prolate spheroid. The variational formula for the reduced problem and its well-posedness are discussed. Boundary element approximation for the variational problem and its error estimates, which have relation to the mesh size and the terms after the series is truncated, are also presented. Two numerical examples are presented to demonstrate the effectiveness and error estimates of this method.

Mathematics subject classification: 65N38, 65N30.
Key words: Natural boundary reduction, Prolate spheroid boundary, Finite element, Exterior harmonic problem.

1. Introduction

Starting from Green's function and Green's formula, natural boundary element method reduces the boundary value problem of partial differential equation into a hypersingular integral equation on the boundary, and then solves the latter numerically [1,15]. Since the variational principle can be conserved after the natural boundary reduction, some useful properties, e.g., self-adjointness and coerciveness, can also be preserved well. Thus the existence, uniqueness and stability of the solution of resulting boundary integral equation can be obtained conveniently. However, it is difficult to obtain Green's functions for most general domains. Therefore the natural boundary element method is very efficient when it is used to solve some exterior boundary value problems and singular problems with a special boundary, such as circle [8,15], ellipse [9,16], and spherical surface [2,6]. But for general cases, only natural boundary element method is not enough, we need the coupling or domain decomposition methods.

The coupling of natural boundary element method and finite element method is applied to solve boundary value problems in general unbounded domains, sometimes for simplicity we still call it as natural boundary element method, or more shortly, DtN method [1,8,10,14,15]. Its basic idea is described as follows. First, the unbounded domain is divided into two subregions, a bounded inner region and an unbounded outer one, by introducing an artificial boundary.

* Received March 22, 2005; Final revised October 12, 2005.
1) This work was subsidized by the National Basic Research Program of China under the grant G19990328, 2005CB321701, and the National Natural Science Foundation of China under the grant 10531080.

Next, the original problem is reduced to an equivalent one in the bounded region. There are many ways to accomplish this reduction. However, the advantages of the natural boundary reduction just as described above ensure that the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form, so not only the analysis of the discrete problem is simplified but also the optimal error estimates and the numerical stability are restored [10].

In the three-dimensional unbounded domain, a sphere [2,6] is usually selected as the artificial boundary. However, for elongated cigar-shaped or ship-shaped obstacles, a prolate spheroid boundary can enclose the obstacles very efficiently, since it leads to smaller computational domain. Therefore, in this paper, we study natural boundary reduction for Laplace equation with Dirichlet or Neumann boundary condition in a three-dimensional unbounded domain outside a prolate spheroid. On the basis of the given results in this paper, we will further study the coupling of finite element and natural boundary element and domain decomposition algorithm based on natural boundary reduction. By using the method of separation of variables and spherical harmonic functions, we express the Poisson integral formula and natural integral operator in a series form explicitly. Thus the original problem is reduced to a boundary integral equation on a prolate spheroid. In real calculation, we truncate the series in finite terms. The variational formula for the reduced problem, the concerned formula after truncating and their well-posedness are all discussed. Boundary element approximation for the variational problem and the concerned error estimates are also presented. The truncation error is often ignored in lots of previous papers but appears in [12,13]. Our error estimates are not only based on the mesh size but also on the terms N after truncating. Two numerical examples are presented to demonstrate the numerical method and their error estimates. We may apply the similar method to solving the same problem outside an oblate spheroid boundary.

2. Poisson Integral Formula and Natural Integral Equation

Let $\Gamma_0 = \{(x,y,z) : \frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1, a > b > 0\}$ denote a prolate spheroid and Ω^c be an unbounded domain outside the boundary Γ_0. We consider the following exterior Dirichlet problem:

$$\begin{cases} \Delta u = 0, & in\ \Omega^c, \\ u = u_0, & on\ \Gamma_0, \\ \textit{some conditions at infinity,} \end{cases} \tag{2.1}$$

and the exterior Neumann problem:

$$\begin{cases} \Delta u = 0, & in\ \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = g_0, & on\ \Gamma_0, \\ \textit{some conditions at infinity,} \end{cases} \tag{2.2}$$

where ν denotes the unit exterior normal vector on Γ_0(regarded as the inner boundary of Ω^c), u_0 and g_0 are the known function on Γ_0 for corresponding problem, respectively. From [5], we know if $g_0 \in H^{-\frac{1}{2}}(\Gamma_0)$, problem (2.2) is well-posed in $W^1(\Omega^c)$ and if $u_0 \in H^{\frac{1}{2}}(\Gamma_0)$, problem (2.1) is also well-posed in $W^1(\Omega^c)$, here

$$W^1(\Omega^c) = \{v \in \mathcal{D}'(\Omega^c) : \frac{v}{r}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \in L^2(\Omega^c)\}, \tag{2.3}$$

where $\mathcal{D}(\Omega^c) = \{\ v : v \textit{ infinitely differentiable on } \Omega^c \textit{ and with compact support in } \Omega^c\}$, and