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A LEAST-SQUARES METHOD FOR CONSISTENT MESH TYING

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Abstract. In the finite element method, a standard approach to mesh tying is to apply Lagrange multipliers. If the interface is curved, however, discretization generally leads to adjoining surfaces that do not coincide spatially. Straightforward Lagrange multiplier methods lead to discrete formulations failing a first-order patch test [12]. A least-squares method is presented here for mesh tying in the presence of gaps and overlaps. The least-squares formulation for transmission problems [5] is extended to settings where subdomain boundaries are not spatially coincident. The new method is consistent in the sense that it recovers exactly global polynomial solutions that are in the finite element space. As a result, the least-squares mesh tying method passes a patch test of the order of the finite element space by construction. This attractive computational property is illustrated by numerical experiments.

Key Words. finite elements, mesh tying, least-squares, first-order elliptic systems

1. Introduction

Mesh tying, or domain bridging, is the opposite of substructuring. A substructuring method solves a boundary value problem using subdomains formed by clustering finite elements from a given discretization of a domain Ω. A mesh tying method solves the same problem by using a discretization of Ω, composed of subdomains that were meshed completely independently. The weak problem is obtained by joining subdomain problems through a suitable variational principle. The simplest nontrivial case of mesh tying is as follows. Assume that Ω is an open bounded domain with Lipschitz continuous boundary Γ, composed of two subdomains; $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The *interface* between the two domains, $\sigma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, is a connected, non empty set. We want to solve numerically the elliptic boundary value problem

$$(1) \qquad -\nabla \cdot \mathbf{A}\nabla\varphi + \alpha\varphi = f \text{ in } \Omega, \quad \text{and} \quad \varphi = h \text{ on } \Gamma,$$

using independently defined finite element partitions of Ω_1 and Ω_2, with boundary conditions imposed on each $\Gamma_i = \Gamma \cap \overline{\Omega}_i$ as shown in Figure 1. This computational setting arises in several different contexts. Equations with discontinuous coefficients are ideally formulated as transmission or interface problems with σ aligned to the discontinuity. Another example is solid mechanics in which two deforming bodies come into contact at σ. A third example arises when for practical and efficiency

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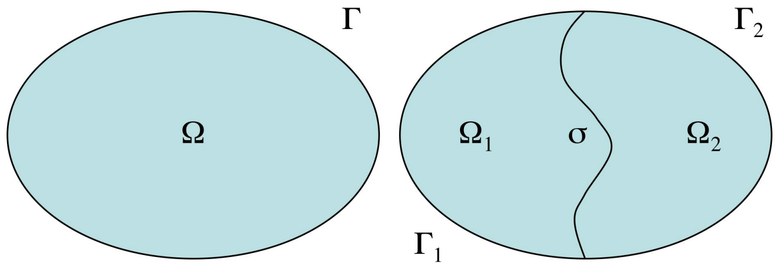

FIGURE 1. The domain Ω is composed of two subdomains, shown on the right.

reasons, grid generation on Ω is replaced by independent meshing of its subdomains. Among other things this approach enables an embarrassingly parallel mesh generation and simplifies meshing of bodies with complex geometries.

1.1. Specifics of mesh tying. In mesh tying Ω is first partitioned into subdomains and then each subdomain is discretized independently. Let Ω_i^h denote a discretization of Ω_i, $i = 1, 2$. The discrete subdomains induce approximations Γ_1^h, Γ_2^h, σ_1^h and σ_2^h of Γ_1, Γ_2 and the interface σ, respectively. Discretization of Ω is given by $\Omega^h = \Omega_1^h \cup \Omega_2^h$. In mesh tying there are two basic configurations for the discrete interfaces σ_1^h and σ_2^h. The first one is when the adjoining surfaces spatially coincide, $\sigma_1^h = \sigma_2^h = \sigma^h$. Typically, this happens when σ is polygonal and can be matched exactly by, e.g., simplexes; see the bottom row in Figure 2. Such interfaces may arise from cutting a complex shape into simpler subdomains to improve efficiency of the mesher. The general case, $\sigma_1^h \neq \sigma_2^h$, typically happens when σ is curved and cannot be represented exactly[1] even by elements with curved sides. This configuration, illustrated in the top row of Figure 2, arises in problems with discontinuous coefficients and contact problems, where the problem definition naturally leads to curved interfaces. In contrast, in domain decomposition and substructuring methods, the discrete domain Ω_h is determined first, and the subdomains are defined *afterwards* as shown in Figure 3. As a result, in these methods the adjoining interfaces always coincide, $\sigma_1^h = \sigma_2^h = \sigma^h$.

A minimal requirement for any mesh-tying or domain bridging method is a consistency condition called *patch test*. A method passes a patch test of order k if it can recover any solution of (1) that is a polynomial of degree k. When $\sigma_1^h \neq \sigma_2^h$ mesh tying methods based on Lagrange multipliers experience difficulties and naively defined schemes fail even a first-order patch test, see [12] for an example. Several approaches have been proposed to address this problem in both two and three dimensions [9, 8, 10, 6, 7, 11, 12]. The methods considered in these papers usually start by selecting one of the non-matching interfaces as a master and the other as a slave surface. The approach of [8, 9, 10] defines Lagrange multipliers on the slave

[1]While finite element methods routinely replace curved boundaries Γ by polygonal approximations Γ^h, the situation is fundamentally different when a curved interface σ is replaced by two spatially distinct discrete interfaces σ_1^h and σ_2^h. While both cases can be viewed as variational crimes in the sense of [14, p.193], the former case leads to a perturbation of the original problem that can be estimated by the Strang's lemma [14, Lemma 4.1, p.186]. For polygonal approximations the error in energy is $O(h^3)$; see [14, p.196]. In the latter case, the discrete computational domain $\Omega_1^h \cup \Omega_2^h$ has gaps and overlaps where the problem ceases to be well-defined. In the overlap regions the 'solution' is multiple valued, and in the voids it is undefined.