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A POSTERIORI ENERGY-NORM ERROR ESTIMATES FOR ADVECTION-DIFFUSION EQUATIONS APPROXIMATED BY WEIGHTED INTERIOR PENALTY METHODS*

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Abstract

We propose and analyze a posteriori energy-norm error estimates for weighted interior penalty discontinuous Galerkin approximations of advection-diffusion-reaction equations with heterogeneous and anisotropic diffusion. The weights, which play a key role in the analysis, depend on the diffusion tensor and are used to formulate the consistency terms in the discontinuous Galerkin method. The error upper bounds, in which all the constants are specified, consist of three terms: a residual estimator which depends only on the elementwise fluctuation of the discrete solution residual, a diffusive flux estimator where the weights used in the method enter explicitly, and a non-conforming estimator which is nonzero because of the use of discontinuous finite element spaces. The three estimators can be bounded locally by the approximation error. A particular attention is given to the dependency on problem parameters of the constants in the local lower error bounds. For moderate advection, it is shown that full robustness with respect to diffusion heterogeneities is achieved owing to the specific design of the weights in the discontinuous Galerkin method, while diffusion anisotropies remain purely local and impact the constants through the square root of the condition number of the diffusion tensor. For dominant advection, it is shown, in the spirit of previous work by Verfürth on continuous finite elements, that the local lower error bounds can be written with constants involving a cut-off for the ratio of local mesh size to the reciprocal of the square root of the lowest local eignevalue of the diffusion tensor.

Mathematics subject classification: 65N30, 65N15, 76R99.
Key words: Discontinuous Galerkin, Weighted interior penalty, A posteriori error estimate, Heterogeneous diffusion, Advection-diffusion.

1. Introduction

In this work, we are interested in a posteriori energy-norm error estimates for a particular class of discontinuous Galerkin (dG) approximations of the advection-diffusion-reaction

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equation

$$\begin{cases} -\nabla\cdot(K\nabla u) + \beta\cdot\nabla u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where for simplicity homogeneous Dirichlet boundary conditions are considered. Here, Ω is a polygonal domain in $\mathbb{R}^d$ with boundary $\partial\Omega$, $\mu \in L^\infty(\Omega)$, $\beta \in [L^\infty(\Omega)]^d$ with $\nabla\cdot\beta \in L^\infty(\Omega)$, $\tilde{\mu} := \mu - \frac{1}{2}\nabla\cdot\beta$ is assumed to be nonnegative, the diffusion tensor K is a symmetric, uniformly positive definite field in $[L^\infty(\Omega)]^{d,d}$ and $f \in L^2(\Omega)$. Owing to the above assumptions, (1.1) is well-posed; see, e.g., [1].

DG methods received extensive interest in the past decade, in particular because of the flexibility they offer in the construction of approximation spaces using non-matching meshes and variable polynomial degrees. For diffusion problems, various DG methods have been analyzed, including the Symmetric Interior Penalty method [2, 3], the Nonsymmetric method with [4] or without [5] penalty, and the Local Discontinuous Galerkin method [6]; see [7] for a unified analysis. For linear hyperbolic problems (e.g., advection–reaction), one of the most common approaches is to use upwind fluxes to formulate the DG method [8, 9]. A unified theory of DG approximations encompassing elliptic and hyperbolic PDEs can be found in [10, 11]. The approximation of the advection-diffusion-reaction problem (1.1) using DG methods has been analyzed in [12] and more recently in [1] with a focus on the high Péclet regime with isotropic and uniform diffusion. The case of high contrasts in the diffusivity poses additional difficulties. Recently, a (Symmetric) Weighted Interior Penalty method has been proposed and analyzed to approximate satisfactorily (1.1) in this situation [13]. The key idea is to use weighted averages (depending on the normal diffusivities at the two mesh elements sharing a given interface) to formulate the consistency terms and to penalize the jumps of the discrete solution by a factor proportional to the harmonic mean of the neighboring normal diffusivities; the idea of using weighted interior penalties in this context can be traced back to [14]; see also [15].

The present paper addresses the a posteriori error analysis of the weighted interior penalty method. Many significant advances in the a posteriori error analysis of dG methods have been accomplished in the past few years. For energy-norm estimates, we refer to the pioneering work of Becker, Hansbo and Larson [16] and that of Karakashian and Pascal [17], while further developments can be found in the work of Ainsworth [18, 19] regarding robustness with respect to diffusivity and that of Houston, Schötzau and Wihler [20] regarding the *hp*-analysis; see also [21, 22]. Furthermore, for L^2-norm estimates, we mention the work of Becker, Hansbo and Stenberg [23], that of Rivière and Wheeler [24], and that of Castillo [25]. Broadly speaking, two approaches can be undertaken to derive a posteriori energy-norm error estimates; in [16, 18, 21], a Helmholtz decomposition of the error is used, following a technique introduced in [26, 27], while the analysis in [17, 20] relies more directly on identifying a conforming part in the discrete solution. The analysis presented herein will be closer to the latter approach. We also mention recent work relying on the reconstruction of a diffusive flux; see [28, 29].

This paper is organized as follows. §2 presents the discrete setting, including the weighted interior penalty bilinear form used to formulate the discrete problem. §3 contains the main results of this work. The starting point is the abstract framework for a posteriori error estimates presented in §3.1 and which is closely inspired by the work of Vohralík for mixed finite element discretizations [30]. Then, §3.2 addresses the case of pure diffusion with heterogeneous and possibly anisotropic diffusivity. We derive an upper bound for the error consisting of three error indicators, i.e. a residual, a diffusive flux and a non-conforming one. This form is similar