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AN ANISOTROPIC NONCONFORMING FINITE ELEMENT METHOD FOR APPROXIMATING A CLASS OF NONLINEAR SOBOLEV EQUATIONS*

Dongyang Shi and Haihong Wang
Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China
Email: shi_dy@zzu.edu.cn, waih777@163.com
Yuepeng Du
Department of Computer, Nanyang Institute of Technology, Nanyang 473004, China
Email: nitduyp@163.com

Abstract

An anisotropic nonconforming finite element method is presented for a class of nonlinear Sobolev equations. The optimal error estimates and supercloseness are obtained for both semi-discrete and fully-discrete approximate schemes, which are the same as the traditional finite element methods. In addition, the global superconvergence is derived through the postprocessing technique. Numerical experiments are included to illustrate the feasibility of the proposed method.

Mathematics subject classification: 65N30, 65N15.
Key words: Nonlinear Sobolev equations, Anisotropic, Nonconforming finite element, Superclosenesss, Global superconvergence.

1. Introduction

Consider the following nonlinear Sobolev equations [1]

$$\begin{cases} -\nabla\cdot\big(a(u)\nabla u_t\big)-\nabla\cdot\big(b(u)\nabla u\big)=f(\mathrm{x},t), & \mathrm{x}\in\Omega, t\in(0,T],\\ u(\mathrm{x},t)=0, & \mathrm{x}\in\partial\Omega, t\in[0,T],\\ u(\mathrm{x},0)=u_0(\mathrm{x}), & \mathrm{x}\in\Omega, \end{cases} \tag{1.1}$$

where $\mathrm{x}=(x,y)$, Ω is a bounded convex domain in R^2, ∇ and $\nabla\cdot$ denote the gradient and the divergence operators, respectively; $a(u)=a(\mathrm{x},t,u)$ and $b(u)=b(\mathrm{x},t,u)$ depend on x, t and u. In (1.1) and below, for notational convenience, we drop the dependence of these coefficients on x and t. Furthermore, we assume that $a(u)$ and $b(u)$ satisfy the following properties as [2]

(i) There exist constants a_0, a_1, b_0 and b_1, such that

$$0<a_0\le a(u)\le a_1,\quad 0<b_0\le b(u)\le b_1. \tag{1.2}$$

(ii) Both $a(u)$ and $b(u)$ are globally Lipschitz continuous in u, i.e., for some constants C_ξ, they satisfy

$$|\xi(u_1)-\xi(u_2)|\le C_\xi|u_1-u_2|,\quad u_1,u_2\in R,\quad \xi=a,b. \tag{1.3}$$

In addition, $a(u)$ and $b(u)$ are twicely continuously differentiable with respective to u.

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It is known that Sobolev equations have important applications including the flow of fluids through fissured rock, the transport problems of humidity in soil, thermodynamics etc. Many studies have been devoted to conforming finite elements. For example, for linear case, [3] considered the first-order generalized difference scheme and gave L^p-norm and $W^{1,p}$-norm error estimates by means of the Ritz-Volterra projection; [4] studied two least-squares Galerkin finite element schemes, which yielded the approximate solutions with optimal accuracy in $(L^2)^2 \times L^2$ norm and the first-order and second-order accuracy in time, respectively; [5] proposed an H^1-Galerkin mixed finite element method and established optimal error estimates for the semi-discrete scheme and fully-discrete scheme. For nonlinear case, [6] gave finite difference streamline diffusion schemes with convection dominated term, and derived the stability and optimal error estimates; [7] considered the time stepping along characteristic finite element methods, and demonstrated optimal convergence rate in the sense of H^1 and L^2; [2] presented discontinuous Galerkin method with penalties and derived $L^\infty(H^1)$ error estimate for the semi-discrete scheme and $L^\infty(H^1)$ and $L^2(H^1)$ for the fully-discrete scheme.

However, there are still some defects in the work mentioned above. On the one hand, although the detailed and systematic theoretical analysis were given in [2-7], there were no numerical tests except [4] in one-dimension. On the other hand, to the best of our knowledge, all the known results in the literature are based on the classical regularity assumption or quasi-uniform assumption on the meshes, i.e., there exists a constant $C > 0$, such that for all element K, $h_K/\rho_K \leq C$ or $h/h_{\min} \leq C$, where $h = \max\limits_K h_K, h_{\min} = \min\limits_K h_K$, h_K and ρ_K are the diameter and the superior diameter of all circles contained in K, respectively (see [8] for details). However, in some cases, the solutions of some elliptic problems may have anisotropic behavior in some parts of the solution domain. This means that the solutions only vary significantly in certain directions. An obvious idea to reflect this anisotropy is to use anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. Besides, some problems may be defined in narrow domain, for example, in modeling a gap between rotator and stator in an electrical machine, the cost of calculation will be very high when the regular partition is employed. Therefore, it is a better choice to employ anisotropic meshes with few degrees of freedom to overcome the above difficulties. Because the anisotropic elements K are characterized by $h_K/\rho_K \to \infty$ when the limit is considered as $h \to 0$, the well-known Bramble-Hilbert lemma can not be used directly in estimating the interpolation error. At the same time, the consistency error estimate, the key of the nonconforming finite element analysis, will become very difficult to be dealt with, for there will appear a factor $|F|/|K| \to \infty$ when the estimate is made on the longer sides F of the element K. It means that the traditional techniques for finite element analysis are no longer valid.

Recently, there have appeared some studies focusing on the study of convergence, supercloseness and superconvergence of anisotropic finite element methods. Both conforming and nonconforming finite elements have been applied to some linear problems, we refer to Acosta [9-10], Apel [11-13], Duran [14] and Shi [15-25]. Whether the results of the above literature are valid for nonlinear problems with anisotropic nonconforming elements remains open.

The purpose of this paper is to apply an anisotropic nonconforming finite element method to (1.1). Firstly, we consider both semi-discrete and backward Euler fully-discrete schemes and obtain the optimal convergence estimates. By virtue of the special property of the element and the postprocessing technique, the supercloseness and superconvergence are obtained. Secondly, we carry out some numerical tests to examine the numerical performance of the element with