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INCREMENTAL UNKNOWNS AND GRAPH TECHNIQUES WITH IN-DEPTH REFINEMENT

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Abstract. With in-depth refinement, the condition number of the incremental unknowns matrix associated to the Laplace operator is $p(d)O(1/H^2)O(|\log_d h|^3)$ for the first order incremental unknowns, and $q(d)O(1/H^2)O((\log_d h)^2)$ for the second order incremental unknowns, where d is the depth of the refinement, H is the mesh size of the coarsest grid, h is the mesh size of the finest grid, $p(d) = \dfrac{d-1}{2}$ and $q(d) = \dfrac{d-1}{2}\dfrac{1}{12}d(d^2-1)$. Furthermore, if block diagonal (scaling) preconditioning is used, the condition number of the preconditioned incremental unknowns matrix associated to the Laplace operator is $p(d)O((\log_d h)^2)$ for the first order incremental unknowns, and $q(d)O(|\log_d h|)$ for the second order incremental unknowns. For comparison, the condition number of the nodal unknowns matrix associated to the Laplace operator is $O(1/h^2)$. Therefore, the incremental unknowns preconditioner is efficient with in-depth refinement, but its efficiency deteriorates at some rate as the depth of the refinement grows.

Key Words. finite differences, incremental unknowns, hierarchical basis, Laplace operator, Poisson equation, Chebyshev polynomials, Fejér's kernel.

1. Introduction

The incremental unknowns—first introduced by Temam [22] through approximate inertial manifolds and spatial multilevel finite-difference discretizations—are a natural tool to study the long-term dynamic behavior of nonlinear dissipative evolutionary equations. Although only dyadic and triadic refinements have been considered so far, Temam has already suggested the use of incremental unknowns with in-depth refinement, ibid., page 169.

As an example, the numerical solution of the incompressible Navier-Stokes equations [20, 21] with Dirichlet boundary value conditions on a staggered marker-and-cell (MAC) grid [16] entails the numerical solution of the (generalized) Poisson equation with Dirichlet and Neumann boundary conditions on a classical and staggered grid [13]; the incremental unknowns with dyadic refinement appear there as an efficient preconditioner. In what follows, we present an analysis of the Poisson equation: we first introduce the equation, then its spatial finite-difference discretization (variational approach), the self-similar interpolating continuous function, the

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incremental unknowns with in-depth refinement and the graph techniques. With $\Omega =]0,1[\times]0,1[$, the Poisson equation with Dirichlet boundary conditions is

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} &= \varphi \quad \text{on } \Gamma = \partial\Omega. \end{cases}$$

We consider the preconditioned incremental unknowns matrix $\mathcal{K}^{-1}\widehat{A}_h$, where $\widehat{A}_h = S^T\widetilde{A}_hS$. Here $\widetilde{A}_h = \mathcal{P}^TA_h\mathcal{P}$, where $\mathcal{P}$ stands for the permutation matrix from hierarchical order to lexicographical order, $A_h = -\boldsymbol{\Delta}_h$, and $\boldsymbol{\Delta}_h$ is the finite-difference Laplace operator. In addition, S stands for the transfer matrix from the incremental unknowns ζ to the nodal unknowns u, i.e., $u = S\zeta$, and $\mathcal{K}$ stands for a suitable symmetric block diagonal matrix.

With in-depth refinement, the condition number of the incremental unknowns matrix associated to the Laplace operator is $p(d)O(1/H^2)O(|\log_d h|^3)$ for the first order incremental unknowns, and $q(d)O(1/H^2)O((\log_d h)^2)$ for the second order incremental unknowns, where d is the depth of the refinement, H is the mesh size of the coarsest grid, h is the mesh size of the finest grid, $p(d) = \dfrac{d-1}{2}$ and $q(d) = \dfrac{d-1}{2}\dfrac{1}{12}d(d^2-1)$. Furthermore, if block diagonal (scaling) preconditioning is used, the condition number of the preconditioned incremental unknowns matrix associated to the Laplace operator is $p(d)O((\log_d h)^2)$ for the first order incremental unknowns, and $q(d)O(|\log_d h|)$ for the second order incremental unknowns. For comparison, the condition number of the nodal unknowns matrix associated to the Laplace operator is $O(1/h^2)$. Therefore, the incremental unknowns preconditioner is efficient with in-depth refinement, but its efficiency deteriorates at some rate as the depth of the refinement grows.

Related conditioning analyses for dyadic refinement are done using a functional analytic argument [4, 3, 2, 24], whereas here we present a purely linear algebraic reasoning for in-depth refinement, following the corresponding analysis with dyadic refinement from [12].

This analysis consists in:

- describing the block-matrix structure of the matrix $(S\mathcal{K}^{-1}S^T)^{-1}$, with graph techniques;
- deriving an appropriate upper bound of the preconditioned generalized Rayleigh quotient
$$\frac{(v,(S\mathcal{K}^{-1}S^T)^{-1}v)}{(v,h^2(-\boldsymbol{\Delta}_h)v)};$$
- deriving an upper bound of the maximum eigenvalue of the incremental unknowns matrix $\widehat{\mathcal{A}}_h$.

Incremental unknowns with triadic refinement have been introduced by Poullet [19] for the numerical solution of the generalized Stokes equations. Moreover, computational experiments displayed therein (see page 37, Fig. 6) show that this condition number is $O((\log_3 h)^2)$, agreeing with the theoretical results presented herein (with the coarsest grid reduced to one point). No conditioning analysis is reported therein.

As usual, the symbols $(\cdot,\cdot)$ and $|\cdot|$ will denote the scalar product and norm of the Hilbert space $L^2(\Omega)$. Throughout this article, c will denote an absolute positive constant, which may be different at different occurrences.

The outline of this paper is as follows. In Section 2, we present the incremental unknowns framework: first we introduce the incremental unknowns with in-depth