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NONLINEAR RANK-ONE MODIFICATION OF THE SYMMETRIC EIGENVALUE PROBLEM*

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Abstract

Nonlinear rank-one modification of the symmetric eigenvalue problem arises from eigenvibrations of mechanical structures with elastically attached loads and calculation of the propagation modes in optical fiber. In this paper, we first study the existence and uniqueness of eigenvalues, and then investigate three numerical algorithms, namely Picard iteration, nonlinear Rayleigh quotient iteration and successive linear approximation method (SLAM). The global convergence of the SLAM is proven under some mild assumptions. Numerical examples illustrate that the SLAM is the most robust method.

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1. Introduction

We consider a nonlinear rank-one modification of the symmetric eigenvalue problem of the form

$$\left[A + s(\lambda)uu^T\right] x = \lambda x, \tag{1.1}$$

where A is a real symmetric matrix, u is a real column vector and $s(\lambda)$ is a real-valued continuous and differentiable function. The problem (1.1) is an extension of the well-known rank-one modification of symmetric eigenvalue problem $(A+\rho uu^T)x = \lambda x$, where ρ is a real constant [1,2]. The nonlinear rank-one modification problem (1.1) arises from the study of eigenvibrations of mechanical structures with elastically attached loads [3, 4], and calculation of the propagation modes of a circular optical fiber [5, 6].

In section 2 of this paper, we study the existence of eigenvalues of (1.1) under proper assumptions of the function $s(\lambda)$. An interlacing property between eigenvalues of (1.1) and the symmetric matrix A is given. Three numerical algorithms are presented in section 3. In particular, the global convergence of the SLAM is established. In section 4, we compare numerical

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performance of three algorithms for examples ranging from model problem to applications in mechanical structure analysis and fiber optic design. These numerical examples illustrate that the SLAM is the most robust method to solve the nonlinear rank-one modification problem (1.1).

2. Existence of Eigenvalues

Let us first recall the following two well-known theorems which describe the interlacing property between the eigenvalues of the symmetric matrix A and its rank-one updating matrix $A + \rho uu^T$, where ρ is a scalar.

Theorem 2.1. ([2], [7, p.442]) *Suppose that the diagonal entries of $D = diag(d_1, d_2, \ldots, d_n)$ are distinct and ordered such that $d_1 < d_2 < \cdots < d_n$. Assume the components of the vector u are nonzero. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $D + \rho uu^T$. Then if $\rho > 0$, $d_1 < \lambda_1 < d_2 < \lambda_2 < \cdots < d_n < \lambda_n$, and if $\rho < 0$, $\lambda_1 < d_1 < \lambda_2 < d_2 < \cdots < \lambda_n < d_n$.*

Theorem 2.2. ([2], [7, p.397]) *Let $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ be the eigenvalues of A and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $A + \rho uu^T$. Then if $\rho > 0$,*

$$\mu_1 \le \lambda_1 \le \mu_2 \le \lambda_2 \le \cdots \le \mu_n \le \lambda_n, \tag{2.1}$$

and if $\rho < 0$,

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \lambda_n \le \mu_n. \tag{2.2}$$

The following lemma shows that eigenvalue of $A + \rho uu^T$ is an increasing function of ρ.

Lemma 2.1. *Let λ_k and θ_k be the kth eigenvalues of symmetric matrices $A + \rho uu^T$ and $A + \tau uu^T$, respectively. If $\rho \ge \tau$, then $\lambda_k \ge \theta_k$.*

Proof. Note that $A+\rho uu^T$ can be written as a symmetric rank-one modification of $A+\tau uu^T$: $A + \rho uu^T = A + \tau uu^T + E$, where $E = (\rho - \tau)uu^T$. By Weyl's monotonicity theorem, see for example [8, p.203], we have $\lambda_k \ge \theta_k + \lambda_{\min}(E) \ge \theta_k + 0 = \theta_k$. □

Let us now turn to studying the existence of eigenvalues for the nonlinear rank-one modification eigenvalue problem (1.1). We begin with a special case of the form

$$\left[D + s(\lambda)uu^T\right] x = \lambda x, \tag{2.3}$$

where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $d_1 < d_2 < \cdots < d_n$. Furthermore, the components u_i of u are nonzero.

Lemma 2.2. *(a) If $s(d_i) = 0$, then d_i is an eigenvalue of* (2.3)*. (b) If $s(d_i) \ne 0$, then d_i is not an eigenvalue of* (2.3)*.*

Proof. The statement (a) is obvious. The statement (b) can be proven by contradiction. If d_i is an eigenvalue of (2.3), then we have $[D + s(d_i)uu^T]x = d_i x$. By Theorem 2.1, the eigenvalues of $D + s(d_i)uu^T$ are strictly interlaced by eigenvalues of D. This leads to the contradiction $d_i < d_i$. □

Based on Lemma 2.2, for the simplicity of exposition, we assume $s(d_i) \ne 0$ for $i = 1, 2, \ldots, n$ for the rest of the paper.