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MULTI-LEVEL ADAPTIVE CORRECTIONS IN FINITE DIMENSIONAL APPROXIMATIONS *

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Abstract

Based on the Boolean sum technique, we introduce and analyze in this paper a class of multi-level iterative corrections for finite dimensional approximations. This type of multi-level corrections is adaptive and can produce highly accurate approximations. For illustration, we present some old and new finite element correction schemes for an elliptic boundary value problem.

Mathematics subject classification: 65B05, 65D15, 65J05, 65N15, 65N30.
Key words: Adaptive, Boolean sum, Correction, Finite dimensional, Multi-level

1. Introduction

Our multi-level corrections are based on the Boolean sum technique. The idea of applying the Boolean sum technique to construct highly accurate finite dimensional approximations may be dated back to [22, 23], in which some local two-level and three-level finite element correction schemes were derived. In this paper, we shall propose a type of multi-level iterative corrections for finite dimensional approximations. This type of schemes is adaptive and is proposed to produce highly accurate approximations based on some simple postprocesses.

Let us give a little more detailed description of the main idea. Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space and A and B be two operators on $\mathcal{H}$. It is known that the so-called Boolean sum of A and B is defined by $A \oplus B = A + B - AB$. It is easy to see that

$$I - (A \oplus B) = (I - A)(I - B).$$

Hence as an operator from a subspace of $\mathcal{H}$ to another subspace, there may hold that

$$\|I - (A \oplus B)\| < \|I - B\|$$

for some proper operator A, which is the key that motivates our multi-level corrections. More precisely, let $u \in \mathcal{H}$ and Bu be an approximation to u. Then $(A \oplus B)\,u$ may be a better approximation than Bu for some simple operator A, where both Au and ABu are computable in application. Note that the construction of A is associated with some subspace of $\mathcal{H}$ and the Boolean sum technique in the multi-level correction in this paper is indeed a successive subspace correction approach (see Section 2 for details).

We should mention that the Boolean sum technique has been applied to design efficient numerical schemes in approximation theory, numerical integration, numerical partial differential

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and numerical integral equations, etc., see, e.g., [2, 4, 5, 8–11, 13, 15–18, 20, 25–27, 29, 35–37, 39, 40, 43] and references therein. We refer to [28, 42] for other interesting connections.

Throughout this paper, we shall use the letter C (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences. For convenience, the symbol $\lesssim$ will be used in this paper. The notation that $x_1 \lesssim y_1$ means that $x_1 \leq Cy_1$ for some positive constant C that is independent of mesh parameters.

2. Multi-Level Correction

We shall discuss the multi-level corrections in a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ that can be compactly embedded into an inner product space $(\mathcal{H}, <\cdot,\cdot>)$, where associated norms are $\|\cdot\|$ and $|\cdot|$, respectively.

Let $\mathcal{K}$ be an operator on $\mathcal{H}$ defined by

$$(\mathcal{K}w, v) = < w, v > \quad \forall w \ \ \forall v \in \mathcal{H}.$$

Then $\mathcal{K}$ is compact on $(\mathcal{H}, \|\cdot\|)$. Let $\mathcal{V} \subset \mathcal{H}$ be a finite dimensional subspace of $\mathcal{H}$ and $P_{\mathcal{V}} : \mathcal{H} \longrightarrow \mathcal{V}$ be a projection operator (namely $P_{\mathcal{V}}^2 = P_{\mathcal{V}}$) satisfying

$$\|u - P_{\mathcal{V}}u\| \lesssim \inf\{\|u - v\| : v \in \mathcal{V}\} \ \ \forall\ u \in \mathcal{H}. \tag{2.1}$$

Set

$$\rho_{\mathcal{V}} = \sup_{u\in\mathcal{H},\|u\|=1} |u - P_{\mathcal{V}}u|. \tag{2.2}$$

Then

$$\begin{aligned} |u - P_{\mathcal{V}}u| &\lesssim \rho_{\mathcal{V}}\|u\|, && \forall\ u \in \mathcal{H}, \\ |u - P_{\mathcal{V}}u| &\lesssim \rho_{\mathcal{V}}\|u - P_{\mathcal{V}}u\|, && \forall\ u \in \mathcal{H}. \end{aligned} \tag{2.3}$$

Consequently,

$$\inf\{|u - v| : v \in \mathcal{V}\} \lesssim \rho_{\mathcal{V}} \inf\{\|u - v\| : v \in \mathcal{V}\}, \ \ \forall\ u \in \mathcal{H}. \tag{2.4}$$

Lemma 2.1. *There hold*

$$\rho_{\mathcal{V}} \lesssim \left(\|(I - P_{\mathcal{V}})\mathcal{K}\| + \|\mathcal{K}(I - P_{\mathcal{V}})\|\right)^{1/2}, \tag{2.5}$$

$$\lim_{\mathcal{V}\to\mathcal{H}} \rho_{\mathcal{V}} = 0, \tag{2.6}$$

where $\mathcal{V} \to \mathcal{H}$ *means that*

$$\inf_{v\in\mathcal{V}} \|u - v\| \to 0 \ \ \forall\ u \in \mathcal{H}. \tag{2.7}$$

Proof. We divide the proof into four steps. First, note that for any $u \in \mathcal{H}$, there hold

$$\begin{aligned} &|(I - P_{\mathcal{V}})u|^2 \\ =&(\mathcal{K}(I - P_{\mathcal{V}})u, (I - P_{\mathcal{V}})u) \\ =&(\mathcal{K}(I - P_{\mathcal{V}})u - P_{\mathcal{V}}\mathcal{K}(I - P_{\mathcal{V}})u, (I - P_{\mathcal{V}})u) + (P_{\mathcal{V}}\mathcal{K}(I - P_{\mathcal{V}})u, (I - P_{\mathcal{V}})u), \end{aligned}$$