Journal of Computational Mathematics
Vol.30, No.5, 2012, 544–554.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1203-m3813

A P-VERSION TWO LEVEL SPLINE METHOD FOR SEMI-LINEAR ELLIPTIC EQUATIONS*

Xinping Shao   Danfu Han   Xianliang Hu
Deptartment of Mathematics, Zhejiang University, Hangzhou 310027, China
Email: shaoxinping1983@163.com mhdf2@zju.edu.cn xlhu@zju.edu.cn

Abstract

A novel two level spline method is proposed for semi-linear elliptic equations, where the two level iteration is implemented between a pair of hierarchical spline spaces with different orders. The new two level method is implementation in a manner of p-adaptivity. A coarse solution is obtained from solving the model problem in the low order spline space, and the solution with higher accuracy are generated subsequently, via one step Newton or monidifed Newton iteration in the high order spline space. We also derive the optimal error estimations for the proposed two level schemes. At last, the illustrated numerical results confirm our error estimations and further research topics are commented.

Mathematics subject classification: 65N30, 65M55.
Key words: P-version, Two level method, Spline methods, Semi-linear, Error estimation.

1. Introduction

The finite element methods are widely used for its convenience and efficiency in the construction of the finite element spaces. Practically, any spline space can be viewed and used as the finite element space [7, 13]. Several applications appeared in numerical solution for Partial Differential Equations (PDEs) in recent years. Lai and Wenston [6] suggest a spline method for steady state Navier-Stokes equation in spline spaces, where the Newton iteration is employed to resolve the nonlinearity. Speleers and Dierckx [12], Li and Wang [8] prefer constructing such spline space in explicit manners. At the same time, Awanou and Lai [1, 2] generalize their method to three dimensional cases, which is also referred as spline element method. Their research illustrates that the general definition for the spline space is so flexible that it is convenient in many complex cases, as well as fulfilling the high order and/or high smoothness requirements in finite element applications.

It is also well accepted that one can obtain high resolution when the mesh size is small or the degree of the spline space is large enough, however, it could be rather time and storage consuming. Two level methods can effectively reduce the costs arising from the refinement of the finite element spaces. In many applications, the two level methods are implemented via mesh refinement, hence it is also named as two-grid methods. It has been proved by Xu et. al [9,14,15] that the two-grid methods have optimal convergence rates for finite element solution of the semi-linear elliptic equations. Recently, we developed a two-grid method for the spline methods [11], where the two grid acceleration is proved to be effective in the circumstance of spline methods.

* Received July 11, 2011 / Revised version received March 16, 2012 / Accepted March 28, 2012 / Published online September 24, 2012 /

On the other hand, it is true that the spline method with local p-refinement, which means raising the degree of spline space locally, can reduce the costs in the linear cases [5], however, the local p-adaptivity is sensitive and some parameters have to be adjusted when applying to different problems. In the current research, we are interested in the spline methods with global p-refinement, and the two level method is considered to reduce the costs arising from global p-refinement. It is potentially more competitive than mesh refinement strategy and we refer it as a p-version two level spline method.

The rest of this paper is organized as following. In the subsequent section, we explain the details of the new two level spline methods for the semi-linear elliptic equation, and the corresponding optimal error estimations are derived in Section 3. Numerical examples are illustrated in Section 4, where the results verify the accuracy of the proposed schemes. Some conclusions and remarks are figured out in the last section. For the ease of reference, let us introduce some basic notations first. Denote $W_p^m(\Omega)$ as the standard Sobolev space equipped with p-norm $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ is the corresponding semi-norm. As the traditional settings, $W_2^m(\Omega) := H^m(\Omega)$ and $H_0^1(\Omega)$ is referred as its restriction with zero boundary. We also assume $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$ as usual.

2. The P-version two Level Spline Methods

2.1. The spline spaces

Let $\triangle$ be a regular triangulation. Define a bivariate spline space $S_d^r(\triangle)$ on $\triangle$ as

$$S_d^r(\triangle) := \Big\{ s \in C^r(\Omega), s|_t \in \mathbf{P}^d, t \in \triangle \Big\},$$

where $\mathbf{P}^d$ is the space of bivariate polynomials with degree d and smoothness $r \geq 0$. Such spline space $S_d^r(\triangle)$ with smoothness r exists provided that the degree $d \geq 3r+2$. For the proof of the existence, we refer to [7].

There are different representation forms of a polynomial with degree d. In the implementation of the multivariate spline spaces, the B-form is preferred to use. Let a triangle $t \in \triangle$ with vertices (v_1, v_2, v_3) and $(\lambda_1, \lambda_2, \lambda_3)$ be the barycentric coordinates of any point (x, y) with respect to triangle t, then any polynomial p with degree d can be represented by

$$p := \sum_{i+j+k=d} c_{ijk} B_{ijk}(\lambda_1, \lambda_2, \lambda_3), \tag{2.1}$$

where $\{c_{i,j,k}\}_{i+j+k=d}$ are called B-coefficients and $B_{ijk}(\lambda_1, \lambda_2, \lambda_3) = \dfrac{d!}{i!j!k!}\lambda_1^i \lambda_2^j \lambda_3^k$ are the bivariate Bernstein Polynomials for all $i+j+k=d$.

Now we consider the global C^r smoothness conditions for a spline on Δ. As a matter of fact, the C^1 case is enough here, and one can refer to [7] for more general C^r cases. It is sufficient to consider the case crossing the common edge between the adjacent patches. Let the triangle t with vertices v_1, v_2, v_3 and the triangle t' with vertices v_4, v_3, v_2 in $\triangle$ sharing one common edge e. Assume that $\lambda := (\lambda_1, \lambda_2, \lambda_3)$ is the barycentric coordinates of v_4 with respect to t, $\{c_{i,j,k}\}_{i+j+k=d}$ and $\{c'_{i,j,k}\}_{i+j+k=d}$ are the B-coefficients for any spline s on t and t' respectively, then $s \in S_d^1(\triangle)$ if and only if the condition

$$c'_{1,j,k} = \lambda_1 c_{1,j,k} + \lambda_2 c_{0,j+1,k} + \lambda_3 c_{0,j,k+1}, \qquad \forall j+k=d$$