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A ROBUST AND ACCURATE SOLVER OF LAPLACE'S EQUATION WITH GENERAL BOUNDARY CONDITIONS ON GENERAL DOMAINS IN THE PLANE*

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Abstract

A robust and general solver for Laplace's equation on the interior of a simply connected domain in the plane is described and tested. The solver handles general piecewise smooth domains and Dirichlet, Neumann, and Robin boundary conditions. It is based on an integral equation formulation of the problem. Difficulties due to changes in boundary conditions and corners, cusps, or other examples of non-smoothness of the boundary are handled using a recent technique called recursive compressed inverse preconditioning. The result is a rapid and very accurate solver which is general in scope, its performance is demonstrated via some challenging numerical tests.

Mathematics subject classification: 65R20, 65E05.
Key words: Laplace's equation, Integral equations, Mixed boundary conditions, Robin boundary conditions.

1. Introduction

This paper presents techniques for computing highly accurate solutions to Laplace's equation on simply connected domains in the plane using an integral equation formulation of the problem. The solver is capable of handling Dirichlet, Neumann, and Robin boundary conditions on fairly general domains with piecewise smooth boundaries. While such problems can be treated using, for example, the finite element method, there is much to gain by their treatment in an integral equation setting, for example robustness, high accuracy, speed and the dimensionality reduction.

This paper is in a sense a continuation of the paper [9] by Helsing in which Laplace's equation was solved on simply connected smooth domains in the plane with mixed Dirichlet and Neumann boundary conditions. With homogeneous Dirichlet or Neumann boundary conditions, and for smooth domains and smooth boundary data, the sought single or double layer density is itself smooth, but this is no longer true for mixed boundary conditions, even for smooth domains, since singularities or other non-polynomial-like behaviour tends to plague the density near the points where the boundary conditions change. In [9], these difficulties were resolved with very good results using recursive compressed inverse preconditioning first introduced in [12].

In applications, however, one is very likely to encounter domains which are only piecewise smooth, for example a domain may have corners. As is the case with the change in boundary conditions, corners also tend to introduce non-polynomial like behaviour of the layer density. The methods of Bremer and Rokhlin [4] and Bremer, Rokhlin and Sammis [5] produce impressive

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results for such problems with homogeneous boundary conditions, but as a further difficulty, it is not uncommon for boundary conditions to change at these corners. This paper will address such domain/boundary condition configurations, and in addition, we will also implement Robin, or third kind boundary conditions. The perhaps most well known application of such boundary conditions is Newton's law of cooling in heat transfer, but there are others in a variety of fields, for example robotic motion [7], chemical vapor deposition modeling [16], and modeling of macromolecular transport in medicine [3].

While the high accuracy property of integral equation solvers is not essential for many practical applications, it is always good to know that the capability of high accuracy is available, and if not needed one may trade some of the accuracy for speed. One might argue that an accuracy of twelve digits in the solution, an accuracy which is well within the reach of the methods of this paper, is a bit over the top in many cases, but for some problems, due to ill-posedness, the end result may be meaningless unless the underlying solver is accurate enough. As an example, when reconstructing harmonic functions from Cauchy boundary data [10], one can easily lose ten digits of accuracy even on smooth domains, demanding very accurate underlying solvers. Such problems, but with corners, can potentially be addressed using the methods of this paper.

The paper is organized as follows. Section 2 introduces the integral equation formulation for the problem, and this integral equation is discretized in Section 3. In Section 4 we discuss the post-processor and in Section 5 the performance of the method in terms of accuracy, robustness and speed is illustrated via some challenging numerical examples. Finally, in Section 6 we discuss possible improvements and future work.

2. The Integral Equation

In the following we will, for convenience and to make notation short, make no distinction between points or vectors in the real plane $\mathbb{R}^2$ and points in the complex plane $\mathbb{C}$. Consider a simply connected domain Ω, with piecewise smooth boundary Γ. Let Γ be comprised of three disjoint boundary segments, Γ_{D}, Γ_{N}, Γ_{R}, describing Dirichlet, Neumann, and Robin boundary conditions respectively. We seek a function $U(z)$, harmonic in Ω, that satisfies :

$$\lim_{\Omega\ni\tau\to z} U(\tau) = f_{\mathrm{D}}(z), \qquad z\in\Gamma_{\mathrm{D}}, \tag{2.1}$$

$$\lim_{\Omega\ni\tau\to z} \frac{\partial U}{\partial n}(\tau) = f_{\mathrm{N}}(z), \qquad z\in\Gamma_{\mathrm{N}}, \tag{2.2}$$

$$\lim_{\Omega\ni\tau\to z} \frac{\partial U}{\partial n}(\tau) + \alpha(z)U(\tau) = f_{\mathrm{R}}(z), \qquad z\in\Gamma_{\mathrm{R}}, \tag{2.3}$$

where $\partial/\partial n$ denotes the derivative with respect to the outward normal, and the bounded functions $f_{\mathrm{D}}(z)$, $f_{\mathrm{N}}(z)$, and $f_{\mathrm{R}}(z)$ is the Dirichlet, Neumann, and Robin data on the corresponding part of the boundary. The parameter $\alpha(z)\geq 0$ is a bounded, real valued function on Γ_{R}.

Along the lines of [9], we now let the solution $U(z), z\in\Omega\cup\Gamma_{\mathrm{N}}\cup\Gamma_{\mathrm{R}}$ be represented by a real density $\mu(z), z\in\Gamma$,

$$U(z) = \frac{1}{\pi}\int_{\Gamma_{\mathrm{D}}} \mu(\tau)\Im\left\{\frac{\mathrm{d}\tau}{\tau-z}\right\} - \frac{1}{\pi}\int_{\Gamma_{\mathrm{N}}\cup\Gamma_{\mathrm{R}}} \mu(\tau)\log|\tau-z|\mathrm{d}|\tau|\,, \qquad z\in\Omega\cup\Gamma_{\mathrm{N}}\cup\Gamma_{\mathrm{R}}. \tag{2.4}$$