Journal of Computational Mathematics
Vol.30, No.3, 2012, 298–310.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1110-m3537

EXPONENTIALLY FITTED LOCAL DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS*

Tao Yu
Department of Mathematics and Physics, Jinggangshan University, Ji'an 343009, China
Email: yutao@jgsu.edu.cn
Xingye Yue
Department of Mathematics, Soochow University, Suzhou 215006, China
Email: xyyue@Suda.edu.cn

Abstract

In this paper, we study the local discontinuous Galerkin (LDG) method for one-dimensional singularly perturbed convection-diffusion problems by an exponentially fitted technique. We prove that the method is uniformly first-order convergent in the energy norm with respect to the small diffusion parameter.

Mathematics subject classification: 65N30.
Key words: Exponentially fitted, Local discontinuous Galerkin method, Convection-diffusion problem.

1. Introduction

In this paper we consider the one-dimensional convection-diffusion problem:

$$\begin{cases} L_\epsilon u := -\epsilon u'' + (au)' = f(x), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where ϵ is a small positive diffusion coefficient and the convection velocity a is positive.

This is a fundamental model problem in computational fluid dynamics. In general, the solution of the problem has a boundary layer at $x = 1$ and the width of the layer is $\mathcal{O}(\epsilon \ln(1/\epsilon))$. When ϵ is big enough, the problem can be solved well by standard finite element methods. But when ϵ is too small, that is to say the problem is convection dominated, the standard finite element methods do not work well, except for the partition step $h < \epsilon$. But it maybe is impossible, since the computing cost is too expensive.

In order to avoid the difficulties, many investigators have resorted to methods based on exponentially-fitted techniques. In [8–10, 18], the authors explored the so-called L spline to solve the problem and gave some uniform error estimates with respect to the small parameter ϵ. However, there are quite a few other techniques developed to treat this problem. We refer to two books focusing on this topic [15, 17]. In [11], the author proposed the interesting tailored finite point method to solve the singular perturbation problem. There are some other papers (see, e.g., [12, 16]) about this method. An upwind finite difference scheme with the grid formed by equidistributing a monitor function is proposed in [14].

* Received October 4, 2010 / Revised version received September 5, 2011 / Accepted October 9, 2011 / Published online May 7, 2012 /

Recently, some authors (see, e.g., [1, 4–7, 19]) applied DG or LDG methods to solve the problem. In [6], the authors analyzed the minimal dissipation LDG method (MD-LDG) for convection-diffusion or diffusion problems. They took the stabilization parameter α to zero on all the inter-element faces except on some part of the Dirichlet boundary to guarantee that the method is well defined. In [4], the authors used LDG method to solve one dimensional time-dependent convection-diffusion problem and obtained some optimal priori error estimates. The numerical result shows that, on a uniform mesh, it can not get the accuracy when the mesh-size h is bigger than the small diffusion parameter ϵ.

In fact, in all the above mentioned works, only piecewise polynomials are used in the approximate finite element spaces and all the error estimates had the form like $||u - u_h|| \leq Ch^{\alpha}||u||_{\beta}$, where u is the exact solution and u_h is the numerical solution. In general, for singularly perturbed problems, the constant C and the Sobolev norm $||u||_{\beta}$ depend on the negative power of the small diffusion parameter ϵ. Therefore, when the mesh-size $h > \epsilon$, this kind of error estimates does not make sense.

As known, one of the advantages of the DG methods is the flexibility with the finite element approximation space. So in [20], the authors used the approximate spaces including non-polynomial functions such as exponentials. With properly selected spaces, they got much more accurate numerical results than only using piecewise polynomial spaces. However there is no theoretical result given on the uniformly convergence of such methods.

In this paper, we will consider a minimal dissipation exponential-fitted LDG method with no penalty involved, i.e. the stabilization parameter α is identically zero everywhere. A first order uniform convergence is obtained in the energy norm as: $\|q - q_h\|_{L^2(0,1)} \leq ch$, with $q = \sqrt{\epsilon}u'$ and q_h the approximation for q. Here 'uniformly' means that the constant $c > 0$ in the above estimate is independent of either the small parameter ϵ and h or the exact solution u. To do so, the ingredient is that only $\|u'\|_{L^1(0,1)}$ is involved in the error estimate. Throughout this paper, the constant c is independent of the parameter ϵ and the exact solution u.

The paper is structured as follows. In Section 2, we review the minimal dissipation LDG method and then present the numerical scheme and the main result on the uniform error estimate, which is proved in Section 3. In Section 4, we show some numerical results.

2. Exponentially Fitted LDG Method

2.1. Review of LDG method

In this subsection, we introduce the minimal dissipation LDG method discussed in [6]. At first, by introducing a new variable, $q = \sqrt{\epsilon}u'$, the problem (1.1) can be rewrite as follows:

$$\begin{cases} (au - \sqrt{\epsilon}q)' = f(x), & x \in (0,1), \\ q + (-\sqrt{\epsilon}u)' = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Let $\{x_{j+\frac{1}{2}}\}_{j=0}^{N}$, $j = 1, \cdots, N$ be a uniform partition of the interval $[0,1]$. Denote by $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, and $h = 1/N$. Multiplying (2.1) by smooth functions v, w, and integrating over