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A New Vector-Valued Padé-Type Approximation in the Inner Space†

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Abstract. A new vector-valued Padé-type approximation is defined by introducing a generalized vector-valued linear functional from a scalar polynomial space to a vector space. Some algebraic properties and error formulas are presented. The expressions of this Padé-type approximants are provided with the generating function form and the determinant form.

Key words: Vector-valued Pade-type approximation; generalized vector-valued linear functional; generating function; generalized inverse; Samelson inverse.

AMS subject classifications: 65D05; 41A21; 15A09

1 Introducton

There are various definitions for vector Padé approximants. In [10], Wynn was led to use the McLeod isomorphism between vectors and some matrices (Clifford numbers) for establishing results for the Padé approximants. This theory was developed by Graves-Morris in [4,5] by using the theory of vector continued fractions. Graves-Morris and Jenkins [6] introduced an axiomatic approach to generalized inverse vector Pad'e approximants ($GIPA$). Gu [7] extended $GIPA$ to the matrix case on the basis of scalar product of matrices. However, $GIPA$ possesses the degree constraint and divisibility constraint. The constraints imply that the method does not construct vector Padé approximants of type $[m/n]$ when n is an odd number. Roberts [8] gave another approach to vector Padé approximants by using Clifford algebra. Salam [9] defined vector Padé-type approximants and vector Padé approximants following the same ideas as in the case of scalar quantities [1,2]. Salam's approach is based on Clifford's algebra structures in theory. It is difficult to compute vector Padé approximants since it deals with multiplication of concerned vectors.

The aim of this paper is to define a new vector-valued Padé-type approximation ($VPTA$) to improve $GIPA$ in [4-6] following the same ideas as Brezinski developed in the scalar case [2], but it is different from Salam's Clifford algebra approach. At the same time it will be shown that our approach can fill the gap of the method of Graves-Morris and Jenkins. In other words, the method in this paper can construct vector Padé approximants of type $[m/n]$ regardless of

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whether n is even or odd. The definition and construction of $VPTA$ are given in Section 2. Some algebraic properties and error formulas are presented in Section 3. On the basis of the orthogonal polynomials, two explicit determinant formulas of $VPTA$ are established in Section 4.

2 Definition and construction of $VPTA$

Let $\mathbf{H}$ be an inner product space with the element $\vec{v} \in \mathbf{C}^d$ and with inner product

$$(\vec{a}, \vec{b}) = \vec{a} \cdot \vec{b} = \sum_{i=1}^{d} a_i b_i,$$

where

$$\vec{a} = (a_1, a_2, \cdots, a_d)^T, \quad \vec{b} = (b_1, b_2, \cdots, b_d)^T \in \mathbf{H}.$$

Here T denotes the transpose of the given vector. The Euclidean norm of the vector $\vec{v}$ is given by

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}^* = \sum_{i=1}^{d} |v_i|^2, \tag{1}$$

where $\vec{v}^*$ denotes the complex conjugate vector of $\vec{v}$.

On the the basis of (1) the generalized inverse (or Samelson inverse) of a vector $\vec{v}$ is defined by

$$\vec{v}^{-1} = 1/\vec{v} = \vec{v}^*/\|\vec{v}\|^2, \quad \vec{v} \neq 0, \quad \vec{v} \in \mathbf{H}. \tag{2}$$

The generalized inverse (2) was used to build $GIPA$ in [4,5].

Let $\mathbf{P}$ denote the vector space of scalar polynomials in one real variable whose coefficients belong to the complex field $\mathbf{C}$, and let $\mathbf{P}_k$ denote the set of elements of $\mathbf{P}$ of degree less than or equal to k. A d-dimensional vector polynomial $\vec{u}(x) = (u_1(x), u_2(x), \cdots, u_d(x))^T$ is said to be of degree k if

$$k = \max\{deg\{u_1(x)\}, deg\{u_2(x)\}, \cdots, deg\{u_d(x)\}\}.$$

Let $\vec{f}(z)$ be a formal vector power series:

$$\vec{f}(z) = \vec{c}_0 + \vec{c}_1 z + \vec{c}_2 z^2 + \cdots + \vec{c}_n z^n + \cdots, \quad \vec{c}_i \in \mathbf{H},\ z \in \mathbf{C}. \tag{3}$$

Let $\varphi : \mathbf{P} \to \mathbf{C}^d$ be a generalized vector-valued linear functional, acting on x and defined by

$$\varphi(x^n) = \vec{c}_n, \quad \vec{c}_n \in \mathbf{H}, \quad n = 0, 1, 2, \cdots. \tag{4}$$

Assume $|xz| < 1$ we may write

$$(1 - xz)^{-1} = 1 + xz + (xz)^2 + \cdots.$$

It follows from (4) that

$$\begin{aligned} \varphi(1 - xz)^{-1} &= \varphi(1 + xz + (xz)^2 + \cdots) \\ &= \vec{c}_0 + \vec{c}_1 z + \vec{c}_2 z^2 + \cdots + \vec{c}_n z^n + \cdots = \vec{f}(z). \end{aligned}$$

Let v be a scalar polynomial of $\mathbf{P}_n$ of degree n:

$$v(z) = b_0 + b_1 z + \cdots + b_n z^n, \tag{5}$$