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Approximation Theorems of Moore-Penrose Inverse by Outer Inverses†

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Abstract. Let X and Y be Hilbert spaces and T a bounded linear operator from X into Y with a separable range. In this note, we prove, without assuming the closeness of the range of T, that the Moore-Penrose inverse T^+ of T can be approximated by its bounded outer inverses $T_n^\#$ with finite ranks.

Key words: Moore-Penrose inverse; outer inverse.

AMS subject classifications: 47A05, 47A58, 65H05

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1 Introduction and preliminaries

Let X and Y be two Hilbert spaces and T a bounded linear operator from X into Y. We use $D(T)$, $N(T)$ and $R(T)$, respectively, to denote the domain, null space and range of T.

Recall that a linear operator $T^\# : Y \mapsto X$ is said to be an outer inverse of T if $T^\# T T^\# = T^\#$. A linear operator $T^+ : Y \mapsto X$ is said to be the Moore-Penrose inverse of T [1], if T^+ satisfies $D(T^+) = R(T) \oplus R(T)^\perp$ and the four Moore-Penrose equations:

$$TT^+T = T, \qquad T^+TT^+ = T^+ \text{ on } D(T^+),$$

$$T^+T = I - P_{N(T)}, \quad TT^+ = P_{\overline{R(T)}} \text{ on } D(T^+),$$

where $P_{(\cdot)}$ is the orthogonal projection onto the subset in the parenthesis.

It is well known that the approximation theory of Moore-Penrose inverse of linear operators plays an important role in various areas of nonlinear analysis and optimization. The approximations of the Moore-Penrose inverse have been studied in the literature such as [1-7]. For an operator with closed range, Z. Ma and J. Ma gave an approximation theorem of the Moore-Penrose inverse by outer inverses with finite ranks [5]. A natural question is whether the Moore-Penrose inverse of an operator with non-closed range can be approximated by its bounded outer inverses. A fundamental distinction between the case of an operator with closed range and the case of an

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operator with non-closed range is that the Moore-Penrose inverse of an operator with non-closed range turns out to be an unbounded operator. Therefore, approximations to such a generalized inverse by bounded operators can converge only in the point-wise sense at best. In this paper, without assuming the closeness of $R(T)$, we give an approximation theorem which asserts that the Moore-Penrose inverse of an operator with separable range can be approximated by its bounded outer inverses with finite ranks. Moreover, because of the stability of the bounded outer inverse [8], our theorems are very useful in computing the Moore-Penrose inverse and in finding the least-square solution of the operator equation.

2 Main results

Theorem 2.1. *Let X and Y be Hilbert spaces and T a bounded linear operator from X into Y with a separable range. For each positive integer n, there exists a bounded outer inverse $T_n^\#$ of T with finite rank n such that*

$$D(T^+) = \left\{ y : \lim_{n\to\infty} T_n^\# y \quad \text{exists} \right\},$$

and if $y \in D(T^+)$, then

$$T^+ y = \lim_{n\to\infty} T_n^\# y.$$

Proof Without loss of generality, we suppose that $R(T)$ is infinite dimensional. Choose a sequence

$$Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots$$

of finite dimensional subspaces of $\overline{R(T)} \subset Y$ with $\dim Y_n = n$ and $\overline{\cup_{n=1}^{\infty} Y_n} = \overline{R(T)} = N(T^*)^\perp$, where T^* is the adjoint operator of T. Let $X_n = T^* Y_n$. Then

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset \overline{R(T^*)} = N(T)^\perp,$$

$\dim X_n = n$ and

$$\overline{\cup_{n=1}^{\infty} X_n} = \overline{R(T^*)}.$$

Indeed,

$$\begin{aligned} \overline{R(T^*)} &= \overline{R(T^*T)} = \overline{T^*(R(T))} = \overline{T^*\overline{R(T)}} = \overline{T^*(\overline{\cup_{n=1}^{\infty} Y_n})} \\ &= \overline{T^*(\cup_{n=1}^{\infty} Y_n)} = \overline{\cup_{n=1}^{\infty} T^* Y_n} = \overline{\cup_{n=1}^{\infty} X_n}. \end{aligned}$$

Let P_n and Q_n denote the orthogonal projectors from Y onto Y_n and from X onto X_n respectively. Put

$$T_n = P_n T.$$

Then T_n is a bounded linear operator with closed range. Also, $N(T_n)^\perp = R(T_n^*) = R(T^* P_n) = X_n$ and $R(T_n) = Y_n$, since $R(T_n)^\perp = N(T_n^*) = N(T^* P_n) = N(P_n) = {Y_n}^\perp$. In order to construct an outer inverse of T, we define $T_n^\# \in B(Y, X)$ as follows:

$$T_n^\# y = \begin{cases} (T_n|_{X_n})^{-1} y, & y \in Y_n, \\ 0, & y \in Y_n^\perp, \end{cases}$$

Thus $T_n^\#$ is a bounded outer inverse of T with $\dim R(T_n^\#) = \dim X_n = n$. In fact, obviously,

$$T_n^\# y = T_n^\# P_n y \quad \text{for} \quad \text{all} \quad y \in Y.$$